Einstein equations in the null quasi-spherical gauge: Progress report

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Abstract

The null quasi-spherical gauge is being used to implement the numerical solution of the full Einstein equations, without any symmetry assumptions. This report describes the structure of the Einstein equations in the NQS gauge, and then outlines the numerical algorithm and specialised techniques being employed. Some preliminary results showing the decay of the Bondi mass of the spacetime are presented.

Keywords: Einstein equations, quasi-spherical, null characteristic hypersurfaces, eth, spin-weighted spherical harmonics.

1. Introduction

At the University of New England, Andrew Norton and I are working on an ARC-sponsored project to numerically evolve the Einstein equations, without any symmetry assumptions. The algorithm exploits an unusual coordinate gauge and this paper describes the background to the gauge and the structure of the resulting form of the Einstein equations. Some of the numerical techniques we are developing are also briefly outlined.

If the spacetime metric admits a group of symmetries (e.g., axial, or spherical), then it has long been recognised that the Einstein equations can simplify dramatically, particularly when the problem of numerical simulation is considered. Spherical symmetry is the most extreme case, since the Einstein equations reduce to a coupled set of ordinary differential equations. If we write the spherically symmetric metric in generalised Schwarzschild coordinates (sometimes called polar coordinates), then the metric may be parameterised by

$$ds^2 = -e^{-2\delta}\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2 \tag{1}$$

where

$$d\sigma^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \tag{2}$$

is the standard metric on S^2 in polar coordinates (ϑ, φ). The full Einstein equations reduce to three equations for $m(r,t)$ and $\delta(r,t)$,

$$\frac{\partial}{\partial r} m \quad = \quad \tfrac{1}{2} r^2 G_{00} \tag{3}$$

$$\frac{\partial}{\partial t} m = \tfrac{1}{2} r^2 e^{-\delta} \left(1 - 2m/r\right) G_{01} \tag{4}$$

$$\frac{\partial}{\partial r} \delta = \tfrac{1}{2} r \left(1 - 2m/r\right)^{-1} G_{11} \tag{5}$$

where the subscripts $0, 1$ refer to the orthonormal basis

$$e_0 = e^{\delta} \left(1 - 2m/r\right)^{-1/2} \partial_t, \quad e_1 = \left(1 - 2m/r\right)^{1/2} \partial_r. \tag{6}$$

Note that these (r, t) coordinates can only be used in the region $r > 2m(r, t)$, exterior to the apparent horizon, where the area function r has spacelike gradient.

Clearly if the source stress energy vanishes then we recover the Schwarzschild spacetime, in accordance with Birkhoff's Theorem — so the value of this formulation is in studying the effects of matter fields. Thus, we assume that the Einstein equations $G_{\alpha\beta} = 8\pi\kappa T_{\alpha\beta}$ are satisfied with stress-energy satisfying the conservation identity $\nabla^{\beta} T_{\alpha\beta} = 0$.

The Bianchi II identities determine the Einstein tensor component G_{22} algebraically (from G_{00}, G_{01}, G_{11} and their first derivatives), and provide the compatibility relation required by the relations (3,4). Consequently it suffices to solve the radial ordinary differential equations (3) and (5) for m, δ with source terms determined from T_{00}, T_{01}, subject to boundary conditions which ensure that (4) is satisfied at one point on each radial line (e.g., at $r = 0$). The remaining equations are then automatically satisfied by virtue of the Bianchi II identities (satisfied by $G_{\alpha\beta}$ since it arises from a metric), and the conservation law satisfied by the stress-energy $T_{\alpha\beta}$.

Axially symmetric spacetimes do not admit such dramatic simplifications, and consequently their theory and numerical behaviour is not as well understood. There are now several independent numerical codes for solving the axially symmetric equations, though it must be said that the main impetus for considering the axial case numerically arises from the fact that the resulting 2+1 equations are within reach of present computer hardware. Because axial spacetimes are expected (and experienced) to only provide weak sources of gravitational radiation, physical interest lies in the more general case of no symmetry; moreover the axial case leads to considerable numerical problems associated with ensuring regularity at the polar axes, which have no intrinsic physical or geometric interest.

Thus the major focus now is on solving the full Einstein equations, without any symmetry assumption. Our approach is to exploit a coordinate condition motivated by the simplicity of the spherically symmetric metric (1), coupled with null (characteristic) hypersurfaces and a spectral resolution using the Newman-Penrose operator $\eth$ (*eth*). The resulting equations are rather simpler than those obtained from other coordinate conditions, and consequently it becomes practicable to attempt a numerical solution on workstations rather than supercomputers. The downside of these advantages is that the technique is restricted to weak and medium strength gravitational fields — strong gravitational interactions (e.g., binary black hole collapse) must be handled by separate codes, which ultimately would be matched to our (exterior) solver.

2. 3D quasi-spherical metrics

The idea underlying the quasi-spherical coordinate choice is most easily seen by considering the class of 3D Riemannian metrics admitting a foliation by metric 2-spheres, with strictly monotone area function r. Using the S^2 isometries to introduce the usual spherical polar coordinates on the 2-spheres $r = const.$, or by considering perturbations of the standard $\mathbf{R}^3$ metric in polar coordinates, which preserve the $r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$ component, we arrive at the metric condition

$$ds^2 = u^2\, dr^2 + (r\, d\vartheta + \beta^1\, dr)^2 + (r\, \sin\vartheta\, d\varphi + \beta^2\, dr)^2, \tag{7}$$

where the lapse $u = u(r, \vartheta, \varphi)$ and the shift vector $\beta^A = \beta^A(r, \vartheta, \varphi)$, $A = 1, 2$, parameterise this class of metrics.

A novel application of this condition is to the construction of 3-metrics of prescribed scalar curvature, and thereby to the construction of solutions of the Hamiltonian constraint. A calculation shows that (7) has scalar curvature R_M,

$$\begin{aligned} \tfrac{1}{2} r^2 u^2 R_M \;\; &= \;\; (2 - \operatorname{div}\beta) u^{-1} (r\partial_r u - \beta^C u_{|C}) - u\Delta u + u^2 - 1 \\ &\quad + r\partial_r(\operatorname{div}\beta) - \beta^C(\operatorname{div}\beta)_{|C} + 2\operatorname{div}\beta - \tfrac{1}{2}(\operatorname{div}\beta)^2 - \tfrac{1}{2}|\beta_{(A|B)}|^2, \end{aligned} \tag{8}$$

where div and Δ are respectively the standard divergence and Laplacian on S^2.

Viewing u as an unknown function and R_M, β as prescribed fields, this equation gives a parabolic partial differential equation for u on $S^2 \times \mathbf{R}_+$. This PDE was extensively analysed in [1], which provides general conditions on R_M, β which ensure global existence (on $S^2 \times \mathbf{R}_+$) and asymptotic flatness, together with either regular axis $r = 0$ or black hole boundary conditions. Two consequences are the existence of a large class of asymptotically flat Cauchy data for the Einstein equations which are metrically flat inside a bounded region, and a simple proof of the Penrose/isoperimetric inequality for black hole data admitting quasi-spherical foliations satisfying $\operatorname{div}\beta = 0$.

The construction generalises easily to include the extrinsic curvature (second fundamental form) and the full constraint equations — with the momentum constraints having the form of coupled ODE and S^2-elliptic equations, with free data and gauge conditions [2].

3. The null quasi-spherical metric

To extend the quasi-spherical condition to four dimensions, we start by requiring the spacetime metric to admit a foliation by isometric 2-spheres, as in 3-dimensions. Introducing the usual spherical polar coordinates (ϑ, φ) and letting (x^1, x^2) denote any coordinates on the space of 2-sphere leaves of the foliation, the metric must take the form

$$ds^2 = A_1 dx^1 + A_2 dx^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2),$$

where $r = r(x^1, x^2)$ is the area parameter of the 2-spheres, and A_1, A_2 are two 1-forms, linear combinations of $dx^1, dx^2, d\vartheta, d\varphi$. We now assume that $r = x^1$ is a coordinate function, (i.e., $dr \neq 0$), call the second orbit coordinate z instead of x^2, and absorb the $d\vartheta, d\varphi$ cross terms to bring the metric to the form

$$ds^2 = A\,dz^2 + 2B\,drdz + C\,dr^2 + (\beta^1 dr + \gamma^1 dz + r\,d\vartheta)^2 + (\beta^2 dr + \gamma^2 dz + r\sin\vartheta\,d\varphi)^2,$$

for some functions $A, B, C, \beta^A, \gamma^A$.

We now introduce a complex notation, which foreshadows the use of a modification of the Newman-Penrose *eth* operator, and which leads to some simplification in the structure of the Einstein equations. Thus, define the complex fields

$$\beta = \tfrac{1}{\sqrt{2}}(\beta^1 - \mathrm{i}\,\beta^2) \tag{9}$$

$$\gamma = \tfrac{1}{\sqrt{2}}(\gamma^1 - \mathrm{i}\,\gamma^2) \tag{10}$$

and the complex-valued 1-form

$$\theta = \tfrac{1}{\sqrt{2}}(d\vartheta + \mathrm{i}\,\sin\vartheta\,d\varphi). \tag{11}$$

The metric may be written now as

$$ds^2 = A\,dz^2 + 2B\,drdz + C\,dr^2 + 2\,|\beta\,dr + \gamma\,dz + r\bar{\theta}|^2. \tag{12}$$

We may consider β as representing either an S^2 vector field, i.e.,

$$\beta \sim \beta^1 \frac{\partial}{\partial\vartheta} + \beta^2 \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi},$$

or as a section of the complex line bundle over S^2 of spin 1.

Now we assume that the coordinate function z determines a null foliation — this implies that the metric (12) is degenerate (rank 2) modulo dz. Setting $dz = 0$ in (12) we see that we must have $C = 0$, so finally the null quasi-spherical metric has the form (with a renaming of the metric parameters A, B for convenience)

$$ds^2 = -2u\,dz(dr + v\,dz) + 2\,|\beta\,dr + \gamma\,dz + r\bar{\theta}|^2. \tag{13}$$

To summarise, the NQS coordinates $(z, r, \vartheta, \varphi)$ are characterised by the *ansatz* conditions:

- The 3-surfaces $z = const.$ are null;
- The 2-surfaces $(z, r) = const.$ are isometric to standard 2-spheres of radius r where r is a coordinate ie. $dr \neq 0$;
- The coordinates (ϑ, φ) are standard spherical polar coordinates for these 2-spheres.

Note that (13) has 6 free metric parameters $(u, v, \beta^1, \beta^2, \gamma^1, \gamma^2)$, which corresponds to the degrees of freedom expected in a general metric with all gauge degeneracy removed ($g_{\alpha\beta}$ has 10 degrees of freedom, with the coordinate/diffeomorphism conditions giving 4 degrees of degeneracy). In contrast, the metric (12) should have one coordinate degree of freedom remaining — imposing the null condition on z removes this freedom, but the form (12) can be constrained in other ways, such as requiring z to be a time coordinate with level sets having zero mean curvature (maximal slicing gauge), or requiring $C = 1$ (pseudo-flat gauge).

The residual gauge freedom in the NQS gauge may be understood by studying the quasi-spheres in the standard cone C_0^+ in Minkowski space. Lorentz transformations preserve this cone, and either rotate or boost the quasi-spheres. This suggests that there should be 6 functions of two variables worth of gauge freedom remaining (a Lorentz transformation at each (r, z)), and we will see this degeneracy may be removed by specifying the $\ell = 1$ spherical harmonic coefficients of either β or γ.

It is interesting to compare the NQS metric with the Bondi metric form

$$ds^2 = -\frac{Ve^{2\beta}}{r}\,du^2 - 2e^{2\beta}\,dudr + r^2 h_{ab}(dx^a - U^a\,du)(dx^b - U^b\,du), \tag{14}$$

with $a, b = 2, 3$, $x^2 = \vartheta$, $x^3 = \varphi$ and

$$[h_{ab}] = \begin{bmatrix} e^{2\gamma}\cosh 2\delta & \sinh 2\delta \sin\vartheta \\ \sinh 2\delta \sin\vartheta & e^{-2\gamma}\cosh 2\delta \sin^2\vartheta \end{bmatrix}, \tag{15}$$

where u is now the null coordinate, and $\det h^{ab} = \sin^2\vartheta$.

Both Bondi and NQS are based on a null coordinate, but there are significant differences, with consequent advantages and disadvantages, in the conditions used to determine the coordinate foliations on the null surfaces. In Bondi, the angular coordinates (ϑ, φ) are determined by radial coordinate lines using the outgoing null geodesics (giving the $(\vartheta, \varphi) = const.$ lines) and the asymptotic metric (used to determine the (ϑ, φ) labelling on the outgoing geodesics). The radial coordinate is then determined by

$$r^2 \sin\vartheta = \|d\vartheta \wedge d\varphi\|^{-1}, \tag{16}$$

which ensures that the r-level sets have area $4\pi r^2$. This construction gives a global character to the coordinates r, ϑ, φ, since they are determined using the metric structure out to infinity. In contrast, the NQS foliation of the null surfaces does not depend on the global structure, and it has coordinate freedoms (boosts and rotations of each sphere) which do not arise in Bondi. In NQS the outgoing null geodesic generator is

$$\frac{1}{u}\ell := \frac{1}{u}\frac{\partial}{\partial r} - \frac{\overline{\beta}}{ru}\,D_v - \frac{\beta}{ru}\,D_{\overline{v}}$$

(where $v = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\vartheta} - \mathrm{i}\,\frac{\partial}{\partial\varphi})$), rather than the coordinate generator $\frac{\partial}{\partial r}$ in Bondi coordinates. In effect, the NQS shear vector β is equivalent to the Bondi-Sachs S^2-metric distortion components γ, δ.

The most significant difference from our perspective lies in the angular part of the metric — unlike the twisted form (15), the NQS angular metric $d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$ is well-understood, and allows us to use explicit spherical harmonic expansions and the standard eth operator

$$\eth\eta = \frac{1}{\sqrt{2}}\sin^s\vartheta\left(\frac{\partial}{\partial\vartheta} - \frac{\mathrm{i}}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\left(\sin^{-s}\vartheta\,\eta\right),$$

(acting on a spin-s field η), to represent angular derivatives. This sidesteps the usual problems with instabilities which arise with numerical derivatives at the poles $\vartheta = 0, \pi$.

4. NQS connection

A natural choice of coframe for the metric (13) is

$$\begin{aligned} \eta^1 &= v\,dz + \,dr \\ \eta^2 &= u\,dz \\ \eta^3 &= \overline{\gamma}\,dz + \overline{\beta}\,dr + r\,\theta \\ \eta^4 &= \gamma\,dz + \beta\,dr + r\,\overline{\theta} \end{aligned}$$

with

$$ds^2 = -2\eta^1\eta^2 + 2\eta^3\eta^4 \tag{17}$$

(note that the signature has been reversed, to facilitate later comparisons with the usual NP coefficients). The corresponding vector frame, using a traditional notation, is

$$\begin{aligned} \ell = e_1 &= \frac{\partial}{\partial r} - \frac{\overline{\beta}}{r}\,D_v - \frac{\beta}{r}\,D_{\overline{v}} \;=:\; \mathcal{D}_r \\ n = e_2 &= \frac{1}{u}\,(\mathcal{D}_z - v\mathcal{D}_r) \\ &= \frac{1}{u}\,\frac{\partial}{\partial z} - \frac{v}{u}\,\frac{\partial}{\partial r} + \frac{\overline{\beta}\,v - \overline{\gamma}}{r\,u}\,D_v + \frac{\beta\,v - \gamma}{r\,u}\,D_{\overline{v}} \\ m = e_3 &= \frac{1}{r}\,D_v \\ &= \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\vartheta} - \frac{\mathrm{i}}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \end{aligned}$$

and $\overline{m} = e_4$.

It is convenient to introduce the auxiliary variables $H,\ J,\ K,\ Q,\ Q^-,\ Q^+$ by

$$H = \frac{1}{u}(2 - \operatorname{div}\beta), \tag{18}$$

$$J = v(2 - \operatorname{div}\beta) + \operatorname{div}\gamma, \tag{19}$$

$$K = v\eth\beta - \eth\gamma, \tag{20}$$

$$Q = r\frac{\partial\beta}{\partial z} - r\frac{\partial\gamma}{\partial r} + \gamma + [\beta, \gamma], \tag{21}$$

$$Q^- = \frac{1}{u}(Q - \eth u), \tag{22}$$

$$Q^+ = \frac{1}{u}(Q + \eth u). \tag{23}$$

The resulting connection may be described by either the connection matrix of 1-forms, or their compactified representation as the Newman-Penrose spin coefficients

$$\begin{aligned}
r\,\alpha_{NP} &= \frac{\tau}{2} + \tfrac{1}{4}\overline{Q^+} \\
r\,\beta_{NP} &= -\frac{\tau}{2} + \tfrac{1}{4}Q^+ \\
r\,\gamma_{NP} &= \frac{\mathrm{i}}{4\,u}\left(4\tau\mathrm{Im}(v\beta - \gamma) - v\ \mathrm{curl}\,\beta + \mathrm{curl}\,\gamma\right) + \tfrac{1}{2}u^{-1}\mathcal{D}_r v \\
r\,\epsilon_{NP} &= \frac{\mathrm{i}}{4\,u}\left(-4\tau u\mathrm{Im}\beta + u\ \mathrm{curl}\,\beta\right) + \frac{1}{2u}\mathcal{D}_r u \\
r\,\rho_{NP} &= -\frac{uH}{2} = -\tfrac{1}{2}(2 - \mathrm{div}\,\beta) \\
r\,\sigma_{NP} &= \eth\beta \\
r\,\tau_{NP} &= \tfrac{1}{2}Q^- \\
r\,\kappa_{NP} &= 0 \\
r\,\lambda_{NP} &= u^{-1}\overline{K} \\
r\,\mu_{NP} &= -\frac{J}{2\,u} \\
r\,\nu_{NP} &= u^{-1}\bar{\eth}v \\
r\,\pi_{NP} &= \tfrac{1}{2}\overline{Q^+}.
\end{aligned}$$

Here τ is the connection coefficient on S^2, so $\tau = -\frac{1}{\sqrt{2}}\cot\vartheta$ in the usual framing of S^2 based on the ϑ, φ coordinates.

Note that the NP shear σ_{NP} has potential β, which suggests that β might be used as the "seed" of a solution of the Einstein equations. This will be apparent below.

Alternatively, the connection can be given in terms of the connection 1-forms, which have coefficients (with index convention $\omega_{abc} = g(e_a, \nabla_{e_c} e_b)$)

$$\begin{aligned}
r\,\omega_{\ell n\ell} &= \frac{r\mathcal{D}_r u}{u} \\
r\,\omega_{\ell nn} &= \frac{r\mathcal{D}_r v}{u} \\
r\,\omega_{\ell nm} &= \tfrac{1}{2}Q^+ \\
r\,\omega_{\ell mn} &= \tfrac{1}{2}Q^- \\
r\,\omega_{\ell m\ell} &= 0 \\
r\,\omega_{\ell mm} &= \eth\beta \\
r\,\omega_{m\ell\overline{m}} &= \tfrac{1}{2}uH = \tfrac{1}{2}(2 - \mathrm{div}\,\beta) \\
r\,\omega_{mn\ell} &= \tfrac{1}{2}Q^+ \\
r\,\omega_{mnn} &= u^{-1}\,\eth v
\end{aligned}$$

$$
\begin{aligned}
r\,\omega_{mnm} &= u^{-1}K \\
r\,\omega_{nm\overline{m}} &= \tfrac{1}{2}u^{-1}J \\
r\,\omega_{\overline{m}m\ell} &= \tfrac{1}{2}\left(2\tau\left(\overline{\beta}-\beta\right)+\mathrm{i}\ \operatorname{curl}\beta\right) \\
r\,\omega_{\overline{m}mn} &= \frac{\mathrm{i}}{2\,u}\left(4\tau\mathrm{Im}(v\beta-\gamma)-v\ \operatorname{curl}\beta+\operatorname{curl}\gamma\right) \\
r\,\omega_{m\overline{m}m} &= -\frac{\cot\vartheta}{\sqrt{2}}=\tau.
\end{aligned}
$$

5. Constraints on the Einstein tensor $G_{\alpha\beta}$

The conservation identity $\nabla^\beta G_{\alpha\beta}=0$ may be viewed either as imposing constraints on the admissible $G_{\alpha\beta}$, or as a means of reducing the number of Einstein equations to be solved from 10 to 6. The latter viewpoint was described above when discussing the spherically symmetric equations, and underlies well-known techniques for simplifying the structure of the Einstein equations. For example in the Cauchy problem, it suffices to solve the dynamical equations $G_{ij}=0$, $1\le i,j\le 3$, provided the constraint equations $G_{0\alpha}=0$, $0\le\alpha\le 3$, are satisfied on the initial spacelike hypersurface — the conservation law then ensures that $G_{0\alpha}=0$ is satisfied at all later times, since the quantity $C_\alpha := G_{0\alpha}$ satisfies a first order hyperbolic equation with zero initial conditions [6].

The null characteristic version of this identity is perhaps even more useful, since it leads to radial ordinary differential equations or constraints. If the *hypersurface equations*

$$G_{\ell\ell}=0,\quad G_{\ell n}=0,\quad G_{\ell m}=0,\quad G_{mm}=0 \tag{24}$$

are satisfied, then the four conservation identities in the NP formalism reduce to

$$
\begin{aligned}
D_\ell(G_{nn}) &= (\rho_{NP}+\rho^*_{NP}-2\varepsilon_{NP}-2\varepsilon^*_{NP})G_{nn}+D_mG_{n\bar m}+D_{\bar m}G_{nm} \\
&\quad -\tfrac{1}{2}(\mu_{NP}+\mu^*_{NP})G_{m\bar m}-\tfrac{1}{2}(\tau^*_{NP}-2\beta^*_{NP}-2\pi_{NP})G_{nm} \\
&\quad -\tfrac{1}{2}(\tau_{NP}-2\beta_{NP}-2\pi^*_{NP})G_{n\bar m} && (25)\\
D_\ell(G_{nm}) &= D_{\bar m}G_{m\bar m}+(\pi^*_{NP}-\tau_{NP})G_{m\bar m} \\
&\quad +\sigma_{NP}G_{n\bar m}+(2\rho_{NP}+\rho^*_{NP}-2\varepsilon^*_{NP})G_{nm} && (26)\\
0 &= \tfrac{1}{2}(\rho_{NP}+\rho^*_{NP})G_{m\bar m}. && (27)
\end{aligned}
$$

These equations are easily verified by substituting for the NP Ricci coefficients Φ_{ab}

$$
\begin{gathered}
\Phi_{00}=\tfrac{1}{2}G_{\ell\ell},\quad \Phi_{01}=\tfrac{1}{2}G_{\ell m},\quad \Phi_{02}=\tfrac{1}{2}G_{mm},\\
\Phi_{11}=\tfrac{1}{4}(G_{\ell n}+G_{m\bar m}),\quad \Lambda=\tfrac{1}{12}(G_{\ell n}-G_{m\bar m}),\\
\Phi_{12}=\tfrac{1}{2}G_{nm},\quad \Phi_{22}=\tfrac{1}{2}G_{nn},
\end{gathered}
$$

with $G_{\ell\ell}=G_{\ell m}=G_{mm}=G_{\ell n}=0$ into the Bianchi relations (eg. [5, (1.322 k,i,j)]).

If the equations $G_{nn}=0$ and $G_{nm}=0$ are then satisfied on a hypersurface Σ transverse to the radial geodesics, e.g., $\Sigma=\{r=1\}$, then all the Einstein equations will be satisfied, provided the outgoing geodesics have non-zero expansion, $\rho_{NP} \neq 0$. This is equivalent to the condition

$$\operatorname{div}\beta < 2 \tag{28}$$

which also arose in an analogous situation in the study of 3-dimensional metrics. Geometrically this corresponds to the requirement that the outgoing geodesics are always expanding, i.e., there are no conjugate points.

The conditions $G_{nn} = 0, G_{nm} = 0$ on Σ will be satisfied if the hypersurface equations (24) hold and if the metric parameters are such that Σ is uniformly either timelike or spacelike, and the usual hypersurface constraint equations $G_{\alpha\nu} = 0$, $\alpha = 0, \ldots, 3$, are satisfied on Σ where ν is the unit (spacelike or timelike) normal vector to Σ.

6. The hypersurface equations

Bondi and Sachs showed that the hypersurface equations (24) in Bondi coordinates reduce to a system of coupled ordinary differential equations, which involve only derivatives tangent to the null hypersurfaces. We find the same happens in the NQS parameterisation, and the resulting equations are simple enough (when expressed in the auxiliary variables) to be analysed directly. A computation using Mathematica (and checked with Reduce) yields

$$r\frac{\partial H}{\partial r} = \nabla_\beta H + \left(\tfrac{1}{2}\operatorname{div}\beta - \frac{2|\eth\beta|^2}{2-\operatorname{div}\beta}\right) H - \frac{1}{u} r^2 G_{\ell\ell} \tag{29}$$

$$\begin{aligned} r\frac{\partial Q^-}{\partial r} &= \nabla_\beta Q^- + \nabla_{Q^-}\beta + 2(\beta - H\eth u) - \mathrm{i}\,\eth(\operatorname{curl}\beta) \\ &\quad + (-2 + \operatorname{div}\beta + \mathrm{i}\ \operatorname{curl}\beta)\, Q^- + 2r^2 G_{\ell m} \end{aligned} \tag{30}$$

$$\begin{aligned} r\frac{\partial J}{\partial r} &= \nabla_\beta J - (1 - \operatorname{div}\beta) J + u \\ &\quad - \tfrac{1}{2}u|Q^+|^2 - \tfrac{1}{2}u\ \operatorname{div}(Q^+) - r^2 u G_{\ell n} \end{aligned} \tag{31}$$

$$\begin{aligned} r\frac{\partial K}{\partial r} &= \nabla_\beta K + \left(\tfrac{1}{2}\operatorname{div}\beta + \mathrm{i}\ \operatorname{curl}\beta\right) K - \tfrac{1}{2}\eth\beta J \\ &\quad + \tfrac{1}{2}u\eth Q^+ + \tfrac{1}{4}u(Q^+)^2 + \tfrac{1}{2}r^2 u G_{mm}. \end{aligned} \tag{32}$$

These are the primary equations which we solve numerically, although slightly different parameterisations yield better behaviour near infinity. The structure of the equations lends itself to an iterative, ODE-based approach:

1. begin with the primary field β on a null hypersurface $z = z_0$;
2. solving $G_{\ell\ell} = 0$ gives H, and thus $u = (2 - \operatorname{div}\beta)/H$;
3. solving $G_{\ell m} = 0$ gives Q^-, and thus Q and Q^+;

4. solving $G_{\ell n} = 0$ gives J;

5. solving $G_{mm} = 0$ gives K.

At this point we notice that the definitions of J, K may be rearranged to derive the equations

$$\mathcal{D}_\beta \gamma := \eth\gamma + \frac{\eth\beta}{2 - \operatorname{div}\beta} \operatorname{div}\gamma = J \frac{\eth\beta}{2 - \operatorname{div}\beta} - K, \tag{33}$$

$$v = \frac{J - \operatorname{div}\gamma}{2 - \operatorname{div}\beta}, \tag{34}$$

where (33) gives an elliptic system for γ, with source term constructed from J, K. The ellipticity of (33) requires (in addition to the already assumed (28)) that the multiplier term

$$B := \frac{\eth\beta}{2 - \operatorname{div}\beta} \tag{35}$$

satisfy $|B|_\infty < 1$; if $|B|_\infty < 1/\sqrt{3}$ then it can be shown that (33) has 6-dimensional kernel, corresponding to a perturbation of the 6 $\ell = 1$ spin 1 spherical harmonics which form the kernel of the operator $\gamma \mapsto \eth\gamma$. In this case the corresponding vector fields are spanned by the generators of rotations and of conformal motions (boosts) of S^2, and form a representation of the Lorentz algebra $so(3,1)$. In the 3-dimensional Riemannian setting, this algebra also arises explicitly as the shear vector fields arising from any quasi-spherical coordinate system on Euclidean space [3]. We interpret this ambiguity in γ as a coordinate freedom, which allows us to specify the "rest frame" at any (r, z).

The formal solution algorithm thus concludes

6. solve the elliptic system (33) for γ using some condition to fix the kernel term, (e.g., set the $\ell = 1$ terms in the spectral expansion of γ to zero), and use (34) to determine v;

7. determine $\frac{\partial\beta}{\partial z}$ from β, γ and the definition (21)

$$\frac{\partial\beta}{\partial z} = \frac{1}{r} Q + \frac{\partial\gamma}{\partial r} + \frac{1}{r}(\nabla_\gamma \beta - \nabla_\beta \gamma - \gamma);$$

8. evolve β to the next null hypersurface.

Note the central role played by the shear potential β. The Robinson-Trautman metrics may be written in NQS coordinates, and have vanishing shear, $\eth\beta = 0$. The interpretation of the RT metrics as having only purely outgoing gravitational radiation suggests again that β (or more precisely, the $\ell \geq 2$ components of β) represents the incoming radiation. However, at best this is only a heuristic, since the analysis of the linearised Einstein equations in the NQS gauge in [10] showed that the odd part of $\gamma - 2v_0\beta$ (and not β) is a gauge-invariant quantity in the linearised limit.

7. Compatibility (evolution) equations

The Einstein components G_{nn}, G_{nm} by symmetry will provide equations along the ingoing null geodesics, which translate into $\frac{\partial}{\partial z}$ constraint equations. Although these should be satisfied by virtue of initial conditions and the conservation identity, numerically we must expect some divergence from zero. Consequently these equations, analogous to the constraint equations familiar from the Cauchy problem, will provide a measure of the numerical accuracy of an evolution algorithm based on the above outline. Explicitly we have

$$\begin{aligned} r\,\mathcal{D}_z\,(J/u) &= r\,v^2\,\mathcal{D}_r\left(\frac{J}{u\,v}\right) + \frac{J^2}{2\,u} - \frac{J\,v}{u} \\ &\quad + \frac{2\,K\,\bar{K}}{u} - \nabla_{Q^+}v - \Delta v + r^2 u\,G_{nn} \end{aligned}$$

$$\begin{aligned} r\,\mathcal{D}_z Q^+ &= r\,v\,\mathcal{D}_r Q^+ + 2\,r\mathcal{D}_r \eth v + \frac{2\,r\mathcal{D}_r u\,\eth v}{u} \\ &\quad + Q^+\,(J - i\,v\ \mathrm{curl}\,\beta + i\ \mathrm{curl}\,\gamma) \\ &\quad + 2\left(v\beta - \gamma - u^{-1}J\,\eth u - \mathrm{div}\,\beta\,\eth v\right) - \mathrm{i}\,\left(v\,\eth\,\mathrm{curl}\,\beta - \eth\,\mathrm{curl}\,\gamma\right) \\ &\quad - v\,\nabla_{Q^+}\beta + 2\,\nabla_{\eth v}\beta + \nabla_{Q^+}\gamma - 2r^2 u\,G_{nm} \end{aligned}$$

Since $G_{\ell n}$ and G_{mm} are invariant under the interchange $\ell \leftrightarrow n$, they also yield compatibility equations:

$$\begin{aligned} r\,\mathcal{D}_z(uH) &= r\,\mathcal{D}_r(uvH) - u + u^2 H^2\,v - uvH + uH\,\mathrm{div}\,\gamma \\ &\quad + \tfrac{1}{2}u\,|Q^-|^2 - \tfrac{1}{2}u\ \mathrm{div}\,Q^- + u\,r^2 G_{\ell n} \end{aligned}$$

$$\begin{aligned} r\,\mathcal{D}_z(\eth\beta) &= r\,\mathcal{D}_r(v\eth\beta) + \left(\tfrac{1}{2}J - v - i\,v\ \mathrm{curl}\,\beta + i\ \mathrm{curl}\,\gamma\right)\eth\beta \\ &\quad + \tfrac{1}{2}H\,K\,u - \tfrac{1}{4}u(Q^-)^2 + \tfrac{1}{2}u\,\eth Q^- - \tfrac{1}{2}u\,r^2 G_{mm} \end{aligned}$$

Roughly, H, J are the outward and inward expansion of the 2-spheres (ρ_{NP} and μ_{NP}), and $\eth\beta$ and K are the outward and inward shears (σ_{NP} and $\bar{\lambda}_{NP}$). The general structure of the Einstein equations may then be summarised very crudely (ignoring derivatives of the derived potential v and of u):

1. $G_{\ell\ell}$ yields $D_\ell H$, the outward derivative of the outward expansion H;

2. $G_{\ell m}$ yields $D_\ell Q^-$, where Q^- is essentially the *connecting potential* Q — so named because of the identity

$$ur\mathcal{D}_n(\eth\beta) + r\mathcal{D}_\ell K = \eth Q + \eth\beta\,r\mathcal{D}_r v - \eth\gamma + \mathrm{i}\,(\eth\beta\ \mathrm{curl}\,\gamma - \eth\gamma\ \mathrm{curl}\,\beta) \qquad (36)$$

which relates the outgoing shear $\eth\beta$ and the ingoing shear K (note also that $Q = ru\,g(m,[n,\ell])$ measures the non-integrability of the distribution of 2-planes normal to the quasi-spheres);

3. $G_{\ell n}$ yields either $D_\ell(J)$, the outgoing derivative of the incoming expansion J, or $D_n(uH)$, the ingoing derivative of the outgoing expansion H;
4. G_{mm} yields either $D_\ell K$, the outgoing derivative of the incoming shear, or $D_n \eth\beta$, the incoming derivative of the outgoing shear;
5. G_{nn} yields $D_n J$, the incoming derivative of the incoming expansion;
6. G_{nm} yields $D_n Q^+$, the incoming derivative of the connecting potential Q.

This analysis may be performed instead directly using the usual NP spin coefficient formulae [5].

8. Aspects of the Numerical Methods

The fields H, J (spin-0), $\beta, \gamma, Q, Q^+, Q^-$ (spin-1) and K (spin-2), are represented by either their point values (on a rectangular grid in ϑ, φ coordinates, with respect to the usual vector framing), or by their spectral coefficients with respect to a spherical harmonic decomposition in the angular directions. A fixed basis of real-valued spherical harmonic functions leads to bases for all spin-weighted fields, and in particular, fields with spin 0 (functions), spin 1 (vectors) and spin 2 (traceless symmetric 2-tensors). The details of these representations, and the routines used to transform between the different representations, are described in a separate report by Andrew Norton [8]. The point representation simplifies the computation of products of fields, while the spectral representation may be used to compute terms involving angular derivatives $(\eth, \mathrm{div}, \mathrm{curl}, \Delta)$, and to spectrally filter numerical errors.

The elliptic system (33) is solved in the spectral representation using a conjugate gradient method with preconditioner $\eth^{-1}$, which leads in practice to rapid convergence, provided fields are considered in the spectral representation.

Various methods are being explored for the radial integration, including radial spectral decomposition and Runge-Kutta integrators of various orders. The radial grid can be adjusted to allow the integrator to reach to null infinity, which it is hoped will permit an accurate extraction of the asymptotic metric parameters.

Likewise, the best algorithm for evolving the seed field β is yet to be determined. The simplicity of the equations (29,30,32), and the absence of any established techniques for dealing with hyperbolic equations in transport form (but c.f. [7]), does permit more extravagant possibilities: the example computation described below used a 4-th order Runge-Kutta scheme for the time evolution as well as the radial integration.

9. Hawking and Bondi Mass

The Hawking mass of the $(z, r) = const.$ 2-spheres is defined using the incoming and outgoing expansions ρ_{NP}, μ_{NP}, and in the NQS variables becomes

$$m_H(z, r) = \tfrac{1}{2} r \left(1 - \frac{1}{8\pi} \oint_{S^2} HJ\right) \tag{37}$$

where the integral is over the unit 2-sphere and

$$\oint_{S^2} HJ = \oint_{S^2} \frac{1}{u}(2 - \operatorname{div}\beta)(\operatorname{div}\gamma + v(2 - \operatorname{div}\beta)).$$

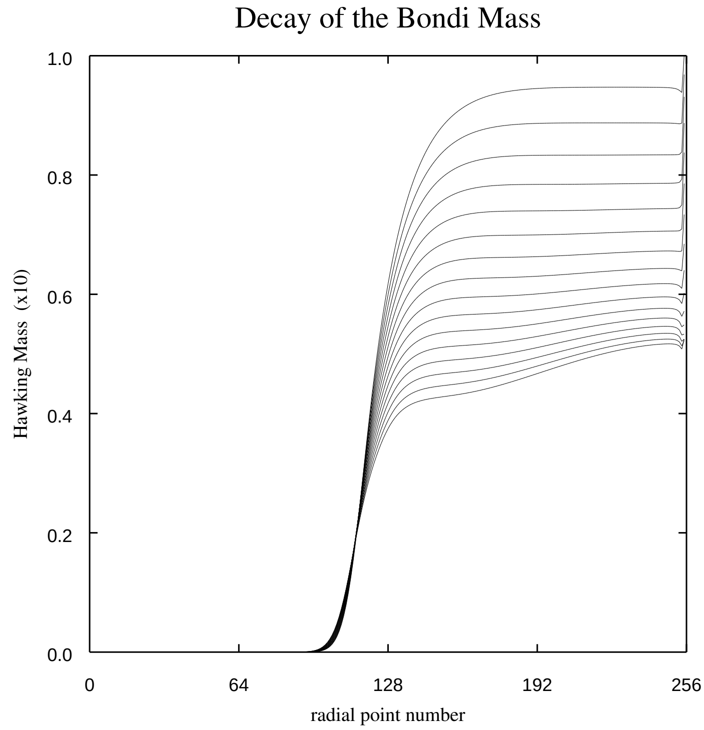

Figure 1: Evolution of the Hawking and Bondi mass functions, for $z = 0 : 7.5$

The Bondi mass of the null hypersurface may be defined as the limit of the Hawking mass at null infinity

$$m_B(z) = \lim_{r \to \infty} m_H(r, z).$$

Figure 1 shows the Hawking mass function at several times in a trial evolution run. I must emphasise that this evolution code has not yet been subjected to rigorous validation tests — these will form an important part of the development over the next year — but the results at this stage are at least promising. The evolution used a radial grid of 256 points, with variable r spacing to resolve the solution near Scri and the final 10 points spanning the interval $17,000 < r < \infty$. The angular grid contained 16×32 points, corresponding to a spherical harmonic resolution of $L = 15$, and the time step was $dz = 0.05$. The initial data was constructed using a test field β given by a random mixture of $l = 2, 3, 4$ spherical harmonics and supported in the region $10 < r < 100$. The numerical calculation was performed on a DEC Alpha at UNE, taking about 6 minutes/timestep and using about 45Mb of memory. The graph was produced by the data visualisation package AVS.

The numerical code produces the functions H, J, from which the Hawking mass function is computed via (37). The graph shows the Hawking mass at z-intervals of 0.5, from $z = 0$ to $z = 7.5$. Note that despite the delicate cancellations involved in (37) near infinity, there is a clearly defined asymptotic decay of m_H to a constant limit, at all times. The spike and dip features which are visible in the final five points ($r > 50,000$), appear to be caused by a combination of this delicate cancellation coming unstuck a little, and accumulated aliasing effects arising from the low $\ell \leq 15$ S^2 spectral cutoff. The later times correspond to the lower Hawking mass curves, so we have a numerical verification of the well-known decay in time of the Bondi mass. Although preliminary, this example does suggest that the NQS technique will lead to a viable exterior integrator for the Einstein equations.

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