Using Riemann normal coordinates in numerical relativity

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Abstract

A new numerical scheme for numerical relativity, based upon Riemann normal coordinates, will be presented. The method will be applied to the construction of initial data for a static Schwarzschild spacetime. It will be shown that the scheme yields second order accurate estimates (in the lattice spacing) for the curvatures of a given metric. However, when used to construct solutions of the constraint equations the method does not appear to produce the correct solution. It is unclear at this stage what the cause of this error is – an error of logic, a numerical error or *quelle horreur* a programming bug.

1. Introduction

The basic idea behind Riemann normal coordinates (see [1,2,3])) is to use the geodesics through a given point to define the coordinates for nearby points. Let the target point be P and consider some nearby point point Q. If Q is close enough to P then there exists a unique geodesic joining P to Q. Let a^μ be the components of the unit tangent vector to this geodesic at P and let s be the geodesic arc length measured from P to Q. Then the Riemann normal coordinates of Q are defined to be $x^\mu = sa^\mu$. These coordinates are well defined provided the geodesics do not cross (we can always choose the neighbourhood of P small enough for this to be true).

An equivalent definition of Riemann normal coordinates at a point P is that they are a set of coordinates for which

$$\Gamma^\mu_{\alpha\beta} = 0 \qquad \text{at } P$$
$$\Gamma^\mu_{\alpha\beta,\nu} + \Gamma^\mu_{\beta\nu,\alpha} + \Gamma^\mu_{\nu\alpha,\beta} = 0 \qquad \text{at } P$$

As a consequence, one immediately obtains, by a Taylor series expansion around P,

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}x^\alpha x^\beta + \mathcal{O}(\epsilon^3)$$

where the coordinates of P have been chosen as $x^\mu = 0$ and ϵ is a typical length scale. The point to note is that the metric is essentially constant (up to quadratic terms) in a small neighbourhood of P. This mathematical contrivance embodies the equivalence principle – that locally spacetime looks flat. It seems like a good idea to build a numerical scheme which at its heart embodies the equivalence principle. This is our main goal.

The construction of a Riemann normal coordinate frame can be performed at any point in the spacetime. However the expansion is valid only in a small neighbourhood of each point. The size of that neighbourhood is limited by the constraint that each point in that neighbourhood must be connected to the origin by a unique geodesic. Its not hard to imagine a large collection of such neighbourhoods being set up so as to provide a complete covering of the spacetime. One question that can now be asked is – How can each neighbourhood (along with its metric) be represented?

The main idea to be presented in this paper is to record this information in a lattice. The vertices of the lattice are just the set of origins of each Riemann normal coordinate frame. The legs of the lattice are the geodesics that join neighbouring vertices. The metric information is contained in the leg lengths and the angles between various legs. One would of course have to be very careful in setting up the lattice so that all of the necessary information (topology, metric and curvature) can be encoded in the lattice.

From the point of view of numerical relativity we would like to propose an inverse construction. Suppose we are given the topology of the lattice (in the form of the connectivity matrix for the vertices) and the leg lengths (and perhaps various angles). Can we then extract the curvature from this information? If the answer is yes then we could demand that these curvatures (and the metric) satisfies Einstein's equations. This must then impose constraints on the original choice of leg lengths. Clearly this amounts to solving Einstein's equations for the metric.

The Regge calculus [4] is another lattice approximation to general relativity. It differs from our proposed new approach in that in the Regge calculus –

- ◇ The metric is not differentiable.
- ◇ The curvature must be viewed as a distribution. It can not be expressed as a point function.
- ◇ The Regge field equations are not a simple transcription of Einstein's equations.
- ◇ Standard tools of analysis such as differentiation are not applicable to the Regge calculus.

In contrast our proposed approach has the following features.

- ◇ The metric is smooth and (at least twice) differentiable.
- ◇ The curvature is a smooth point function.
- ◇ The field equations are exactly the Einstein field equations.
- ◇ All the usual tools of analysis are available.

To emphasize the distinction between a Regge lattice and our lattice we will call our lattice a *smooth lattice*. Perhaps *Riemann lattice* might be a better name.

2. Schwarzschild initial data

As a test of smooth lattice relativity we should be able to successfully recover the time symmetric 3-geometry for the Schwarzschild spacetime, namely

$$ds^2 = \rho(r)^4 \left(dr^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2\right)$$

where $\rho(r) = 1 + m/(2r)$ and r is the isotropic radial coordinate.

Our starting point is to propose a 3-metric in the form

$$ds^2 = \alpha(z)^2 dz^2 + \beta(z)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The coordinate z is a freely chosen radial coordinate. There is only one non-trivial equation

$$0 = {}^{(3)}R$$

which we must solve for α or β. To begin, write the metric in the 2+1 form

$$(g_{\mu\nu}) = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 h_{\mu\nu} \end{pmatrix}$$

where $h_{\mu\nu}$ is the metric of the unit 2-sphere. Then it is a straightforward calculation to show that the above field equation is equivalent to

$$\frac{d}{dl}\left(\frac{1}{\beta}\frac{d\beta}{dl}\right) = \frac{R}{4\beta^2} - \frac{3}{2}\left(\frac{1}{\beta}\frac{d\beta}{dl}\right)^2 \qquad (1a)$$

where $l = \int \alpha dz$ is the proper distance measured from the throat and R is the scalar curvature of the unit 2-sphere.

The boundary conditions at the throat, where $l = 0$, are chosen to be

$$\beta = 1 \qquad \text{and} \qquad \frac{d\beta}{dl} = 0 \qquad (1b)$$

The first condition is equivalent to setting the ADM mass (in fact $\beta = 2m$). The second condition is required for $l = 0$ to be a minimal surface.

Where does the smooth lattice enter into this calculation? We will use it to estimate R. We will do this by constructing a lattice on a 2-sphere and solving the appropriate lattice equations for R.

Since it is well known that $R = 2$ there seems little point in employing a smooth lattice. Though this is a reasonable objection one can take the alternative view – if the method does not work for this highly specialized case then it certainly will be of no use in other situations.

2.1. Smooth lattice 2-sphere

To estimate R it will be necessary to choose a triangulation of the 2-sphere, compute the geodesic leg lengths and finally solve a set of equations (given below) for R.

The simplest approximation to a 2-sphere is a regular tetrahedron (see Fig(1a). Better approximations can be generated by successively sub-diving each triangle according to the pattern in Fig(1b).

The metric of the 2-sphere can be written as

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

or as the induced metric on the surface $1 = x^2 + y^2 + z^2$ in Euclidean 3-space with the usual (x, y, z) coordinates. The (θ, ϕ) are related to the (x, y, z) by the usual polar coordinate transformations. The (x, y, z) coordinates of the four vertices of the original tetrahedron are easily calculated by appealing to the symmetry of the tetrahedron. The coordinates of the vertices of the successive lattices are calculated in a two step process. First each new vertex is introduced to the centre of each old leg. This vertex is then displaced along the radial direction out to the unit sphere. In this way the (θ, ϕ) coordinates of each vertex can be calculated.

The leg lengths for each leg are calculated by solving the geodesic equations as a two point boundary value problem. At the same time we compute $\int ds$ along this geodesic path. This gives us the leg lengths for each geodesic (segment).

2.2. Smooth lattice equations

Suppose each vertex has been assigned some integer label. We will use the notation σ_i to denote the vertex with label i. Now consider the Riemann normal coordinate frame centred on vertex σ_o. Suppose there are n triangles attached to this vertex and that the vertices, starting with σ_o, are labelled, 0 to n.

We are free to choose our Riemann normal coordinates such that $g_{\mu\nu}(x_o) = \mathrm{diag}(1,1)$, $x^\mu_o = 0$ and $x^2_1 = 0$. This exhausts all coordinate freedoms, all the remaining x^μ_i and (one) curvature component(s) must be computed from the given leg lengths.

The metric in this set of triangles is of the form

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} x^\alpha x^\beta + \mathcal{O}(\epsilon^3)$$

It can be shown [2] (by expanding the geodesic equation as Taylor series) that the geodesic joining σ_i to σ_j is described by

$$x^\mu(\lambda) = x^\mu_i + \lambda \Delta x^\mu_{ij} - \frac{\lambda(1-\lambda)}{3} R^\mu{}_{\alpha\nu\beta}\, \Delta x^\alpha_{ij} \Delta x^\nu_{ij}\, x^\beta_i + \mathcal{O}(\epsilon^4)$$

where λ is an affine parameter with $\lambda = 0$ at σ_i and $\lambda = 1$ at σ_j. The geodesic leg length can then be computed as $L_{ij} = \int\, ds$ with the result

$$L^2_{ij} = \left(g_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, \overline{x}^\alpha_{ij}\, \overline{x}^\alpha_{ij} \right) \Delta x^\mu_{ij} \Delta x^\nu_{ij} + \mathcal{O}(\epsilon^6) \tag{2}$$

where $\overline{x}^\mu_{ij} = (x^\mu_i + x^\mu_j)/2$ and $\Delta x^\mu_{ij} = x^\mu_j - x^\mu_i$.

Note that

$$R_{\mu\alpha\nu\beta}\, \overline{x}^\alpha_{ij}\, \overline{x}^\alpha_{ij} \Delta x^\mu_{ij} \Delta x^\nu_{ij} = R_{\mu\alpha\nu\beta}\, x^\mu_i x^\nu_i x^\alpha_j x^\beta_j$$

and further note that, in 2-dimensions, there is only one independent curvature component, which we can take to be R_{1212}.

There are n triangles. Hence we have $2n$ leg lengths L_{ij}. There are also $n+1$ vertices for which there are $2(n+1)$ coordinates x^μ_i to compute. However, we have already chosen 3 of the $2(n+1)$ coordinates. Thus we have to compute $2n-1$ coordinates and one curvature component from the $2n$ leg lengths L_{ij}. Fortunately, we have as many equations as unknowns. (in fact it is easy to see that this will always be true in 2-dimensions provided the surface is fully triangulated).

The $2n$ equations (2) were solved for the x^μ_i and R_{1212} via a Newton-Raphson method. Starting from flat space, the iterations converged in about 3-4 iterations (though more iterations were required for the very coarse approximation of the original tetrahedron).

The estimates so obtained are listed in table (1). Since not every vertex in each approximation is equivalent to every other vertex (they have differing local triangulations) the method returns different estimates for R for each vertex. Hence in the table we have listed the best and worst estimates for R. One can observe that the method converges by a factor of four with each successive sub-division. As the leg lengths are halved with each sub-division this implies the error in R varies as $\mathcal{O}(L^2)$ where L is a typical length scale for leg lengths. In short, the smooth lattice yields 2nd-order accurate estimates for the curvature.

Table 1. Estimates of $|R-2|$ for a unit 2-sphere

Sub-division	Worst estimate	Best estimate	Average estimate
1	1.19	1.19	1.19
2	2.99e-1	4.91e-2	1.71e-1
3	6.40e-2	1.91e-3	2.55e-2
4	1.54e-2	5.67e-5	2.97e-3
5	3.81e-3	5.18e-6	2.47e-4

2.3. Results

Using the worst values for R, from the previous section, the initial value problem (1) was solved using a 4-th order Runge Kutta method starting from the throat and integrating outwards.

The result of the integration is that we have the 3-metric in the form

$$d\tilde{s}^2 = dl^2 + \beta^2(l)d\Omega^2$$

where $d\Omega^2$ is the metric of the unit 2-sphere. We would like to compare this with the known metric

$$ds^2 = \rho^4(r)\left(dr^2 + r^2 d\Omega^2\right)$$

where $\rho(r) = 1 + (m/2r)$. One way to do this is to align the coordinates r and l by integrating, in parallel with the main equation (1),

$$\frac{dr}{dl} = \frac{1}{(1+\frac{m}{2r})^2}$$

starting from $r = m/2$ at $l = 0$. The function $r(l)(\rho(r(l)))^2$ can then be compared directly with $\beta(l)$. The results are shown in Fig(3) and they show a very good agreement. Two sets of graphs have been presented, one for the set of worst estimates of R and the other for a sequence of R values. This later set of graphs clearly shows that the error $\delta\rho$ in ρ, at fixed r appears to vary as $\mathcal{O}(\delta R)$. Thus the global discretization error appears to be $\delta\rho = \mathcal{O}(L^2)$.

3. Discussion

The first point that must be made is that the above test is a very benign test of the smooth lattice method. The sole contribution of the smooth lattice was to aid in the computation of the scalar 2-curvature, which was already known to be 2. One could probably concoct any number of schemes which spit out the magic number 2.

A far better test would be to employ a smooth lattice for the full 3-metric (rather than just the metric of the 2-sphere). Such an investigation has been under way for some time with mixed results. The idea is to sub-divide the 3-space into a large set of cube-like cells with the cells arranged to lie between successive 2-spheres. The edges of the cell that joins a pair of successive 2-spheres are chosen to be segments of the radial geodesics. The symmetry of the space suggests that one tube of cells, extending from the the throat to some large outer region, is sufficient to define the full 3-metric. The results so far show that the curvatures, for a given metric, can be estimated to an order $\mathcal{O}(L^2)$ where L is the typical cell size. In this case the only non-trivial equation, $0 = R$, is also satisfied to order $\mathcal{O}(L^2)$. However, what we really want to do is the reverse, to use the field equation $0 = R$ to construct the 3-metric. We have attempted to do this by starting with the throat and successively adding cubes to the tube. The size of each new cube is constrained by $0 = R$ and the gauge choice that its radial edges are a smooth extension of the radial geodesic. The results are very disappointing – the metric so constructed has an order $\mathcal{O}(1)$ error relative to the correct Schwarzschild metric. At this stage in the investigation it is hard to pinpoint the reason for this error, it could be a bug in the program, it could be a error in the smooth lattice methodology. I would prefer, despite the embarrassment, that the error was a bug in my code! Needless to say much detective work is needed.

The honest assessment of the utility of Riemann normal coordinates, in the form suggested here, for numerical relativity is not that the jury is out but that the jury has yet to hear any substantive evidence. I hope to provide some of that evidence in the near future.

4. Riemann Normal Coordinates – A Summary

For convenience, here are some useful formula for geodesics, angles and so in Riemann normal coordinates. Some of these formulas have already been stated in the body of the paper but they are repeated here just for completeness. It will be assumed for the remainder that $x^\mu = 0$ at the origin of the Riemann normal coordinates.

4.1. Riemann normal coordinates

$$\Gamma^{\mu}_{\alpha\beta} = 0 \qquad \text{at } P$$
$$\Gamma^{\mu}_{\alpha\beta,\nu} + \Gamma^{\mu}_{\beta\nu,\alpha} + \Gamma^{\mu}_{\nu\alpha,\beta} = 0 \qquad \text{at } P$$

4.2. Metric

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} x^{\alpha} x^{\beta} + \mathcal{O}(\epsilon^3)$$

At P,

$$\Gamma^{\alpha}_{\beta\mu,\nu} = -\frac{1}{3}\left(R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\beta\nu}\right)$$
$$g_{\mu\nu,\alpha\beta} = -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right)$$
$$R_{\mu\nu\alpha\beta} = g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta}$$

4.3. Geodesic distance

Let two points have coordinates x^{μ}_i and x^{μ}_j. The geodesic distance between these points is given by

$$\begin{aligned} L^2_{ij} &= (g_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, \overline{x}^{\alpha}_{ij}\, \overline{x}^{\alpha}_{ij}) \Delta x^{\mu}_{ij} \Delta x^{\nu}_{ij} + \mathcal{O}(\epsilon^4) \\ &= g_{\mu\nu} \Delta x^{\mu}_{ij} \Delta x^{\nu}_{ij} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^{\mu}_i x^{\nu}_i x^{\alpha}_j x^{\beta}_j + \mathcal{O}(\epsilon^4) \end{aligned}$$

where $\overline{x}^{\mu}_{ij} = (x^{\mu}_i + x^{\mu}_j)/2$ and $\Delta x^{\mu}_{ij} = x^{\mu}_j - x^{\mu}_i$.

4.4. Geodesics

The geodesic starting from σ_i and passing through to σ_j is described by

$$x^{\mu}(\lambda) = x^{\mu}_i + \lambda \Delta x^{\mu}_{ij} - \frac{\lambda(1-\lambda)}{3} R^{\mu}{}_{\alpha\nu\beta}\, \Delta x^{\alpha}_{ij} \Delta x^{\nu}_{ij}\, x^{\beta}_i + \mathcal{O}(\epsilon^4)$$

where λ is an affine parameter with $\lambda = 0$ at σ_i and $\lambda = 1$ at σ_j.

At $\lambda = 0$

$$\frac{ds}{d\lambda} = L_{ij}$$
$$\frac{dx^{\mu}}{ds} = \frac{1}{L_{ij}} \left(\Delta x^{\mu}_{ij} - \frac{1}{3} R^{\mu}{}_{\alpha\nu\beta} \Delta x^{\alpha}_{ij} \Delta x^{\nu}_{ij} x^{\beta}_i \right) + \mathcal{O}(\epsilon^4)$$

The geodesic passing through σ_i and with unit tangent vector m^{μ} at σ_i is described by

$$x^{\mu}(s) = x^{\mu}_i + s m^{\mu} + \frac{s^2}{3} R^{\mu}{}_{\alpha\nu\beta}\, m^{\alpha} m^{\nu} x^{\beta}_i + \mathcal{O}(\epsilon^4)$$

where s is the geodesic arc length.

4.5. Cosine law

Given a triangle with vertices σ_i, σ_j and σ_k the angle at vertex σ_k can be found from

$$2L_{ik}L_{jk}\cos\theta_k = L_{ik}^2 + L_{jk}^2 - L_{ij}^2 - \frac{1}{3}R_{\mu\alpha\nu\beta}\,\Delta x_{ik}^{\mu}\Delta x_{ik}^{\nu}\Delta x_{jk}^{\alpha}\Delta x_{jk}^{\beta} + \mathcal{O}(\epsilon^5)$$

References

[1] A.Z. Petrov, "*Einstein Spaces*" (Pergamon Press, Oxford, 1969) Chp. 1 Sec. 7.

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