Remarks on the Yilmaz and Alley papers

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Abstract

The claims by Yilmaz and Alley that general relativity does not give the correct results for the attraction between thin planes shells and thin spherical shells are investigated and found to be incorrect.

1. Introduction

In recent papers Yilmaz [16] and Alley [1] have called into question the validity of Einstein's 1915–1916 theory of gravitation, the general theory of relativity. In particular, they have asserted that Einstein's theory is essentially a 1–body test particle theory of gravitation, that it does not contain 2–body solutions, and that it does not give correct answers for the gravitational attraction between two infinite plane slabs of matter or between two concentric shells of matter (Yilmaz[16, p 116]). These claims are in striking contradiction with quite well–known calculations in general relativity, but the relevant papers are not cited by either Yilmaz or Alley. For example, Horský [8] has shown that thin parallel shells of incoherent matter *are* attracted towards each other in hyperbolic motion in general relativity. For the case of a spherical shell in general relativity the fundamental equations have been given by Gerlach [5], including the case where the shell has internal pressure. It is a straightforward calculus exercise to show that Gerlach's results reduce to the well–known Campbell–Hénon [3, 6] results in the Newtonian limit. The additional equations necessary to deal with the case in general relativity where there is an interaction between a number of spherically symmetric concentric shells of matter have been given by Fackerell [4] in an extension to general relativity of the Campbell–Hénon method for spherical clusters. Consequently there are good grounds to look very carefully at the calculations in the papers by Yilmaz and Alley. We first examine their slab calculations.

2. Errors in slab calculation by Yilmaz and Alley

A key part of the Yilmaz–Alley claim that the general theory of gravitation is incorrect is centered in their assertion that in general relativity, plane slabs do not attract one another. Yilmaz asserts in his paper that the plane–symmetric metric

$$ds^2 = -e^{-2\phi}c^2\,dt^2 + e^{2\phi}\left[e^{2\epsilon\phi}(dx^2+dy^2) + e^{4\epsilon\phi}\,dz^2\right],$$

where $\epsilon = \pm 1$ and ϕ has the form $\phi = Az + \frac{1}{2}\sigma z^2 + C$, is a solution of the Einstein gravitational field equations for plane slabs with a matter density σ (note that we adopt the conventions of Misner, Thorne and Wheeler [11], except that we shall put in the relevant powers of c). There is a prima facie case that that Yilmaz's assertion cannot be true, since his claimed solution has the slabs at fixed positions, and, as is well known (c.f. Horský), the slabs would have to be supported in their fixed positions by unphysical stresses. But in any case, let us calculate and check Yilmaz's assertion.

In performing calculations, it is very important not to lose sight of the fact that there is a well–defined procedure for finding solutions in general relativity:-

1. A metric (or equivalently a tetrad) with an appropriate symmetry and therefore appropriate general dependence upon the coordinates is chosen, along with an appropriate energy–momentum tensor $T_{\mu\nu}$, e.g., for vacuum, electrovac, pressure–free dust, or for a perfect fluid with a given equation of state.

2. The assumed metric or tetrad is then used to compute the Einstein tensor $G_{\mu\nu}$.

3. The Einstein field equations
$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$$
are solved in each region.

4. Finally the solutions in the different regions are joined together using the junction conditions
$$K^{+}_{ij} - K^{-}_{ij} = \frac{8\pi G}{c^2}\left(S_{ij} - \frac{1}{2}\,{}^{(3)}g_{ij} S\right),$$
where S_{ij} is the surface energy–momentum tensor of the surface layer between the different regions. Of course, if there is no surface layer, $S_{ij} = 0$ and the relevant condition is continuity of the extrinsic curvature tensor. This condition is not necessarily equivalent to continuity of the normal derivative of some metric coefficient, as Yilmaz and Alley seem to claim in their papers.

Unfortunately, as we shall see, Yilmaz and Alley have not carried out this program in their discussions of general relativity. Instead they appear to have imposed on general relativity the ideas of Yilmaz's new theory. Because of this, in a number of significant cases, what they claim as a solution in general relativity is no solution at all.

With this warning in mind, we check Yilmaz's calculation by taking the orthonormal tetrad

$$\begin{aligned}
\omega^0 &= e^{-\phi} c\, dt, \\
\omega^1 &= e^{(1+\epsilon)\phi} dx, \\
\omega^2 &= e^{(1+\epsilon)\phi} dy, \\
\omega^3 &= e^{(1+2\epsilon)\phi} dz.
\end{aligned}$$

We find that, in the case where $\epsilon = 1$, the non–zero orthonormal tetrad components of the Einstein tensor are given by

$$\begin{aligned} G_{00} &= -4e^{-6\phi}\phi''(z), \\ G_{11} &= e^{-6\phi}\phi''(z), \\ G_{22} &= e^{-6\phi}\phi''(z), \\ G_{33} &= 0. \end{aligned}$$

Consequently, if we are dealing with incoherent matter, we must have $G_{11} = G_{22} = 0$, so that $\phi''(z) = 0$, and this implies that $G_{00} = 0$, so that the density of matter is zero. On the other hand, if we require G_{00} to be non–zero, we find that there must be transverse stress of a non–physical kind. In any case, it is simply not true that the matter density σ is connected to ϕ by the formula given by Yilmaz.

A similar problem arises in the case where $\epsilon = -1$. In that case we have by simple calculation the following tetrad components of the Einstein tensor:-

$$\begin{aligned} G_{00} &= 0, \\ G_{11} &= -e^{2\phi}\phi''(z), \\ G_{22} &= -e^{2\phi}\phi''(z), \\ G_{33} &= 0. \end{aligned}$$

In this case of necessity the matter density has to be zero, whereas a transverse stress is still possible.

The error in Yilmaz's paper [16] seems to go back to a 1979 publication [15] in which he makes calculations relating to plane–symmetric situations in general relativity. There he claims (correctly) that the source–free Einstein equations for the plane–symmetric metric

$$ds^2 = e^{-2\phi}c^2\,dt^2 - e^{\mu}(dx^2 + dy^2) - e^{\eta}\,dz^2,$$

where ϕ, μ and η are functions of z alone, give rise to the equations

$$\begin{aligned} \frac{\mu'(3\mu' - 2\eta')}{8} + \frac{\mu''}{2} &= 0, \\ \frac{4\phi'^2 + (\mu' - 2\phi')(\mu' - \eta')}{8} + \frac{\mu'' - 2\phi''}{4} &= 0, \\ \frac{\mu'(\mu' - 4\phi')}{8} &= 0 \end{aligned}$$

(Actually Yilmaz's paper does not have the "$= 0$" of the last equation, but this is necessary for source–free space).

Yilmaz then goes on to make the assertion that these equations imply that either $\mu = 4\phi$ and $\eta = 6\phi$ or $\mu = 0$ and $\eta = -2\phi$. It turns out that this statement is true if and only if ϕ is a linear function of z. However, in the problem with which Yilmaz and Alley are concerned in their later papers, $\phi'(z)$ is not a constant, so that it is invalid to assume that either $\mu = 4\phi$ and $\eta = 6\phi$ or $\mu = 0$ and $\eta = -2\phi$. Consequently their assumed form of the metric is not correct.

To prove these statements let us look at the general solution of Yilmaz's equations. From the last equation we readily have the result that either $\mu' = 0$ or $\mu' = 4\phi'$, so that after a trivial integration and absorption of a constant of integration we can correctly assert that either $\mu = 0$ or $\mu = 4\phi$. In the case where $\mu = 0$, the first equation is satisfied identically and the second equation becomes

$$2\phi'^2 + \phi'\eta' - 2\phi'' = 0,$$

which, if $\phi' \neq 0$, has as its solution

$$\eta = 2\log|\phi'| - 2\phi + C,$$

where C is a constant. Only in the case $\phi'(z) =$ constant is it possible to choose the constant of integration to make $\eta = -2\phi$.

The situation is no better in the case where $\mu = 4\phi$. For then the two remaining equations become identical, namely,

$$\phi'\eta' = 2\phi'' + 6\phi'^2$$

which has the solution, for $\phi' \neq 0$,

$$\eta = 2\log|\phi'| + 6\phi + C.$$

Again only in the case where $\phi'(z) =$ constant is it possible to choose the constant of integration to make $\eta = 6\phi$.

Since in his later work on plane slabs Yilmaz has $\phi''(z) \neq 0$, it is no longer valid to assume, as he does, that either $\mu = 4\phi$ and $\eta = 6\phi$ or $\mu = 0$ and $\eta = -2\phi$. Consequently we have to conclude that the form of the metric adopted by Yilmaz for plane symmetry is incorrect. Alley also has similar incorrect statements about general relativity in the section of his paper where he claims to calculate the general relativistic solution for two parallel plane slabs. Note also that Alley's metric is not consistent with a correct stress–energy tensor, i.e., one that gives a positive mass density σ without unphysical transverse stresses. This is not surprising, because in a pure gravitational problem the plane slabs would have to move towards one another.

3. Correct general relativistic plane slab calculations

There are a considerable number of calculations that have been carried out in general relativity for situations involving plane slabs of matter, either of incoherent dust or of compressible matter with a prescribed equation of state. Notable among these are the various calculations by A.H. Taub [13] and by J. Horský [7, 9] (neither cited by Yilmaz or Alley). However, it is not without interest to calculate directly the interaction of two thin plane shells of matter, since this shows up most clearly the errors in the statements by Yilmaz and Alley.

The way to handle thin shells of matter in general relativity has been discussed with great clarity by Israel [10]. The regions between the shells are empty space metrics. We join these empty space solutions by the Synge–O'Brien junction conditions. For the empty space metric with plane symmetry, that is, admitting the three Killing vectors ∂_x, ∂_y and $y\partial_x - x\partial_y$, we take the form

$$ds^2 = e^{2\alpha}(-c^2dt^2 + dz^2) + e^{\beta}(dx^2 + dy^2),$$

where α and β are functions of t and z.

Now Taub [12] has demonstrated that the general empty space solution of these equations may be written as

$$ds^2 = \frac{1}{\sqrt{1+kz}}(-c^2dt^2 + dz^2) + (1+kz)(dx^2 + dy^2),$$

where k is a constant. Note carefully however that the coordinate transformations involved in writing the solution in this form may involve the introduction of comoving coordinates for the problem. An example of this occurs in the next Section. We now suppose that we have two empty regions with such symmetry, separated by a thin plane shell of matter with surface density σ and pressure p.

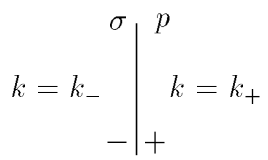

Figure 1: Thin shell of surface density σ and surface pressure p.

In general the k values will be different in the empty regions on the two sides of the plane shell. Since the coordinates x and y are Killing–defined, we may identify x and y on the two sides of the shell. The metric on one side, denoted by $+$, will therefore be

$$ds^2 = \frac{1}{\sqrt{1+k_+z}}(-c^2dt^2 + dz^2) + (1+k_+z)(dx^2 + dy^2),$$

and on the other side, denoted by $-$, the metric will be

$$ds^2 = \frac{1}{\sqrt{1+k_-z}}(-c^2dt^2 + dz^2) + (1+k_-z)(dx^2 + dy^2).$$

Following Israel, we need now to introduce intrinsic coordinates in the shell. The most convenient choice is

$$\begin{aligned} \xi^0 &= \tau, && \text{proper time following shell,} \\ \xi^1 &= x, && \text{Killing defined coordinate,} \\ \xi^2 &= y, && \text{Killing defined coordinate,} \end{aligned}$$

since all these coordinates are continuous across the shell.

Next we consider the induced metric in the shell. Let the parametric equations of the + side of the shell be given by

$$\begin{aligned} z &= Z_+(\tau), \\ t &= T_+(\tau) \end{aligned}$$

and let the parametric equations of the $-$ side of the shell be given by

$$\begin{aligned} z &= Z_-(\tau), \\ t &= T_-(\tau). \end{aligned}$$

Then from the + side, the induced metric is

$$\begin{aligned} ds^2 &= -\frac{1}{\sqrt{1+k_+Z_+}}\left(c^2dT_+^2 - dZ_+^2\right) + (1+k_+Z_+(\tau))\left[(d\xi^1)^2 + (d\xi^2)^2\right] \\ &= -c^2d\tau^2 + (1+k_+Z_+(\tau))\left[(d\xi^1)^2 + (d\xi^2)^2\right]. \end{aligned}$$

From the $-$ side, the induced metric is

$$\begin{aligned} ds^2 &= -\frac{1}{\sqrt{1+k_-Z_-}}\left(c^2dT_-^2 - dZ_-^2\right) + (1+k_-Z_-(\tau))\left[(d\xi^1)^2 + (d\xi^2)^2\right] \\ &= -c^2d\tau^2 + (1+k_-Z_-(\tau))\left[(d\xi^1)^2 + (d\xi^2)^2\right]. \end{aligned}$$

Because of the requirement that the induced metric be the same for both imbeddings, we must have

$$1 + k_+Z_+(\tau) = 1 + k_-Z_-(\tau).$$

Moreover we clearly have

$$\dot{T}_+^2 - \frac{\dot{Z}_+^2}{c^2} = \sqrt{1 + k_+Z_+(\tau)}$$

and

$$\dot{T}_-^2 - \frac{\dot{Z}_-^2}{c^2} = \sqrt{1 + k_-Z_-(\tau)}.$$

If say $k_- = 0$ but $k_+ \neq 0$, we must have $Z_+(\tau) = 0$. If $k_+k_- \neq 0$ we have

$$Z_-(\tau) = \frac{k_+Z_+(\tau)}{k_-}.$$

The next step is to calculate the extrinsic curvature tensor K_{ij} defined by

$$K_{ij} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{n}}{\partial \xi^j}$$

where the tangent vectors $\mathbf{e}_i$ to the shell are defined by

$$(\mathbf{e}_i)^\alpha = e^\alpha_{(i)} = \frac{\partial x^\alpha}{\partial \xi^i}$$

and $\mathbf{n}$ is the unit normal to the shell (Note again that we are using the sign choice of Misner, Thorne and Wheeler). We have

$$\begin{aligned} e^\alpha_{(0)} &= [\dot{T}(\tau), 0, 0, \dot{Z}(\tau)], \\ e^\alpha_{(1)} &= [0, 1, 0, 0], \\ e^\alpha_{(2)} &= [0, 0, 1, 0] \end{aligned}$$

and the components of the unit normal are

$$n^\alpha = [\frac{\dot{Z}(\tau)}{c^2}, 0, 0, \dot{T}(\tau)].$$

Of course the 4–velocity vector of the shell is given by

$$U^\alpha = e^\alpha_{(0)}.$$

These expressions apply on either side with the appropriate subscript. The non–zero Christoffel symbols for the + side are given by

$$\begin{aligned} \Gamma^t_{zt} &= \frac{-\frac{1}{4}k_+}{1+k_+z}, \\ \Gamma^x_{zx} &= \Gamma^y_{zy} = \frac{\frac{1}{2}k_+}{1+k_+z}, \\ \Gamma^z_{tt} &= \frac{-\frac{1}{4}k_+c^2}{1+k_+z}, \quad \Gamma^z_{zz} = \frac{-\frac{1}{4}k_+}{1+k_+z}, \\ \Gamma^z_{xx} &= \Gamma^z_{yy} = -\frac{1}{2}k_+\sqrt{1+k_+z}. \end{aligned}$$

From this we find that the only non–zero K^+_{ij} are given by

$$K^+_{11} = K^+_{22} = -\frac{1}{2}k_+\dot{T}_+(\tau)$$

and

$$K^+_{00} = \frac{\ddot{Z}_+}{\dot{T}_+} - \frac{\frac{1}{4}k_+c^2}{\dot{T}_+\sqrt{1+k_+Z_+}} - \frac{\frac{1}{2}k_+\dot{Z}^2_+}{\dot{T}_+(1+k_+Z_+)}.$$

Israel's fundamental equations for the discontinuity in K_{ij} are

$$K^+_{ij} - K^-_{ij} = \frac{8\pi G}{c^2}\left(S_{ij} - \frac{1}{2}\,{}^{(3)}g_{ij}S\right).$$

Here the intrinsic energy–momentum tensor S_{ij} is given by

$$S_{ij} = (\sigma + \frac{p}{c^2})U_iU_j + {}^{(3)}g_{ij}p$$

where p is the pressure and σ is the surface density of the shell and $S = {}^{(3)}g^{ij}S_{ij}$.

Voorhees [14] has shown that the surface energy–momentum tensor may be found in terms of surface kinetic theory with a distribution function f as

$$S_{ij} = \int \frac{1}{\mu} f p_i p_j \frac{dp_0 \wedge dp_1 \wedge dp_2}{\sqrt{-{}^{(3)}g}}$$

where μ is the rest mass of a typical particle in the surface layer and p_i is the intrinsic momentum of such a particle. We readily find that the intrinsic momentum components of a typical particle in the shell are

$$\begin{aligned} p_1 &= \alpha \cos\psi, \\ p_2 &= \alpha \sin\psi, \\ p_0 &= -c\left[\mu^2 c^2 + \frac{\alpha^2}{1+kZ}\right]^{1/2}, \end{aligned}$$

where α is the magnitude of the transverse momentum and the auxiliary angle ψ ranges over $0 \leq \psi \leq 2\pi$. A useful case is where all of the particles have the same rest mass μ_0 and the magnitude of the transverse momentum α has the constant value α_0, so that the distribution function takes the form

$$f = C\delta(\mu - \mu_0)\delta(\alpha^2 - \alpha_0^2),$$

where C is a constant. This choice gives

$$\begin{aligned} \sigma &= \frac{\pi C}{(1+kZ)}\left(\mu_0^2 c^2 + \frac{\alpha_0^2}{1+kZ}\right)^{1/2}, \\ p &= \frac{\frac{1}{2}\pi C c^2}{(1+kZ)^2} \frac{\alpha_o^2}{\left(\mu_0^2 c^2 + \frac{\alpha_0^2}{1+kZ}\right)^{1/2}}. \end{aligned}$$

Whatever the choice of f, the fundamental equations of motion of the shell are

$$-\frac{1}{2}k_+\sqrt{\dot{Z}_+^2/c^2 + \sqrt{1+k_+Z_+}} + \frac{1}{2}k_-\sqrt{\dot{Z}_-^2/c^2 + \sqrt{1+k_-Z_-}} = \frac{4\pi G}{c^2}(1+kZ)\sigma \qquad (I)$$

and

$$\frac{\ddot{Z}_+}{\dot{T}_+} + \frac{\frac{1}{4}k_+}{\dot{T}_+\sqrt{1+k_+Z_+}} - \frac{\ddot{Z}_-}{\dot{T}_-} - \frac{\frac{1}{4}k_-}{\dot{T}_-\sqrt{1+k_-Z_-}} = -\frac{8\pi G}{c^2}p, \qquad (II)$$

where

$$\dot{T}_\pm = \sqrt{\dot{Z}_\pm^2/c^2 + \sqrt{1+k_\pm Z_\pm}}.$$

4. Case of pressure–free shells

The simplest case, namely, where there are two parallel thin shells of equal surface density σ and zero transverse pressure in the shells, was considered by Horský [8] long ago. Since his treatment is succinct it may be worthwhile to amplify his argument. It turns out that $k = 0$ in the region between the shells so that between the shells we have Minkowski spacetime (this will evident from the validity of the solution obtained).

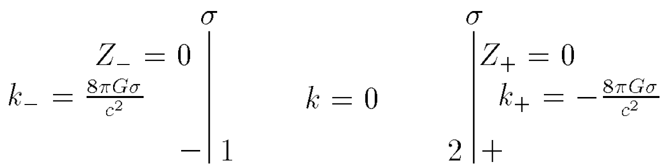

Figure 2: Parallel thin shells of equal surface density σ and zero surface pressure.

Since there are two shells, it is convenient to change our notation slightly, as indicated in the diagram, namely, to label the inside of the right shell as 2 and to label the inside of the left shell as 1, the outside of the left shell being denoted by $-$. Consider the junction conditions for the shell on the right. From the fact that $k = 0$ in the middle, we must have $Z_+(\tau) = 0$. Note that this implies that in the exterior $Z_+ = 0$ is a comoving coordinate for the shell. We then find that equation (I) gives

$$k_+ = -\frac{8\pi G\sigma}{c^2}$$

so that the surface density σ must be a constant and k_+ must be negative. Equation (II) then gives, for $p = 0$,

$$\frac{\ddot{Z}_2}{\sqrt{1 + \dot{Z}_2^2/c^2}} = -2\pi G\sigma,$$

which shows that we do not have comoving coordinates on the inside. In view of the statements by Yilmaz and Alley it needs to be noted that this result reduces, for small velocities, ($|\dot{Z}_2/c| << 1$), to precisely the classical result for the acceleration of the shell of such a shell. The solution is of course that of hyperbolic motion. If we write $2\pi G\sigma = a$, we find that the solution is given by

$$\begin{aligned} Z_2 &= Z_{20} - \frac{c^2}{a}\cosh\left[\frac{a}{c}(\tau_2 - \tau_{20})\right], \\ T_2 &= T_{20} + \frac{c}{a}\sinh\left[\frac{a}{c}(\tau_2 - \tau_{20})\right], \end{aligned}$$

where τ_2 is the proper time on the right–hand shell and τ_{20}, Z_{20} and T_{20} are constants. The solution is completed by setting $k_- = -k_+ = 8\pi G\sigma/c^2$. It is readily found that Z_1 satisfies a similar equation to that satisfied by Z_2, namely,

$$\frac{\ddot{Z}_1}{\sqrt{1 + \dot{Z}_1^2/c^2}} = +2\pi G\sigma,$$

with a corresponding solution, namely,

$$\begin{aligned} Z_1 &= Z_{10} + \frac{c^2}{a}\cosh\left[\frac{a}{c}(\tau_1 - \tau_{10})\right], \\ T_1 &= T_{10} + \frac{c}{a}\sinh\left[\frac{a}{c}(\tau_1 - \tau_{10})\right], \end{aligned}$$

where τ_1 is the proper time on the left–hand shell and τ_{10}, Z_{10} and T_{10} are constants.

5. Spherical Shells

As mentioned in the introduction, the calculations for spherical shells have been carried out by a number of authors. The most general case, for spherical shells of matter with isotropic transverse pressure was carried out by Gerlach [5]. In view of the claims by Yilmaz that the general relativity result for spherical shells does not reduce to the appropriate Newtonian limit, it is perhaps worthwhile spending a little time on this case, particularly since it involves two *finite* bodies. We shall see that Yilmaz's claims concerning spherical shells in general relativity are also false.

Consider a spherically symmetric thin shell of matter, for which the mass outside of the shell is m_2 and the mass inside the shell is m_1 (the mass inside the shell could be due to an interior shell or to a black hole at the centre). In the empty space region outside the shell the metric is

$$ds^2 = -\left(1 - \frac{2Gm_2}{c^2 r}\right) c^2\, dt^2 + \frac{dr^2}{\left(1 - \frac{2Gm_2}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

and for the region between this shell and the next interior concentric shell we have the metric

$$ds^2 = -\left(1 - \frac{2Gm_1}{c^2 r}\right) c^2\, dt^2 + \frac{dr^2}{\left(1 - \frac{2Gm_1}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

Intrinsic coordinates for the shell may be taken as $\xi^0 = \tau$, the proper time on the shell, $\xi^1 = \theta$ and $\xi^2 = \phi$ since the Killing defined coordinates θ and ϕ may be taken to be continuous through the shell. The parametric equations of the outside of the shell are then

$$\begin{aligned} r &= R_+(\tau) \\ t &= T_+(\tau) \end{aligned}$$

and the parametric equations of the inside of the shell are

$$\begin{aligned} r &= R_-(\tau) \\ t &= T_-(\tau). \end{aligned}$$

From the induced metric we find that $R_+(\tau) = R_-(\tau) = R(\tau)$, say. However, Schwarzschild time is not continuous across the shell. From the continuity of the induced metric we find that

$$\frac{dT_+}{d\tau} = \frac{1}{\left(1 - \dfrac{2Gm_2}{c^2 r}\right)} \left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_2}{c^2 R}\right)^{1/2}$$

and

$$\frac{dT_-}{d\tau} = \frac{1}{\left(1 - \dfrac{2Gm_1}{c^2 r}\right)} \left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_1}{c^2 R}\right)^{1/2}.$$

For the components of the tangent vectors to the shell we have on the outside of the shell

$$\begin{aligned} e^\alpha_{(0)} &= [\dot{T}_+(\tau), \dot{R}(\tau), 0, 0], \\ e^\alpha_{(1)} &= [0, 0, 1, 0], \\ e^\alpha_{(2)} &= [0, 0, 0, 1] \end{aligned}$$

and the components of the unit normal are

$$n^\alpha = [\frac{\dot{R}}{f_+ c^2}, f_+ \dot{T}_+(\tau), 0, 0]$$

where $f_+ = \left(1 - \dfrac{2Gm_2}{c^2 R}\right)$. Similar expressions apply to the inside with subscript $-$ and with $f_- = \left(1 - \dfrac{2Gm_1}{c^2 R}\right)$. The non–zero Christoffel symbols for the m_2 side are given by

$$\begin{aligned} \Gamma^t_{rt} &= \frac{Gm_2}{f_+ c^2 R^2}, \\ \Gamma^\theta_{r\theta} &= \frac{1}{r}, \quad \Gamma^\theta_{\phi\theta} = -\sin\theta\cos\theta, \\ \Gamma^\phi_{r\phi} &= \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta, \\ \Gamma^r_{tr} &= \frac{Gm_2}{f_+ c^2 R^2}, \quad \Gamma^r_{rr} = -\frac{Gm_2}{f_+ c^2 R^2}, \quad \Gamma^r_{tt} = \frac{f_+ G m_2}{R^2}, \\ \Gamma^r_{\theta\theta} &= -f_+ R, \quad \Gamma^r_{\phi\phi} = -f_+ R \sin^2\theta. \end{aligned}$$

where $f_+ = \left(1 - \dfrac{2Gm_2}{c^2 R}\right)$. Similar expressions hold for the m_1 side. The non–zero components of the K^+_{ij} are given by

$$K^+_{11} = \frac{K^+_{22}}{\sin^2\theta} = -Rf_+\dot{T}_+$$

and

$$K^+_{00} = \frac{\ddot{R} + \dfrac{Gm_2}{c^2 R}}{f\dot{T}_+}.$$

For shells of particles that all have the same rest mass μ_0 and squared angular momentum α_0^2, the surface density σ and the pressure p are found to be given by

$$\sigma = \frac{A}{4\pi R^2 c}\left[\mu_0^2 c^2 + \alpha_0^2/R^2\right]^{\frac{1}{2}}$$

and

$$p = \frac{A\alpha_0^2 c}{8\pi R^4}\left[\mu_0^2 c^2 + \alpha_0^2/R^2\right]^{-\frac{1}{2}}.$$

The equations of motion are then found to be given by

$$\left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_1}{c^2 R}\right)^{\frac{1}{2}} - \left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_2}{c^2 R}\right)^{\frac{1}{2}} = \frac{4\pi G}{c^2} R\sigma$$

with the conservation equation

$$\frac{d}{d\tau}\left(R^2\sigma\right) + 2R\dot{R}\frac{p}{c^2} = 0.$$

Furthermore,

$$\frac{dT_+}{d\tau} = \frac{1}{\left(1 - \frac{2Gm_2}{c^2 R}\right)}\left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_2}{c^2 R}\right)^{\frac{1}{2}}$$

and

$$\frac{dT_-}{d\tau} = \frac{1}{\left(1 - \frac{2Gm_2}{c^2 R}\right)}\left(\frac{\dot{R}^2}{c^2} + 1 - \frac{2Gm_1}{c^2 R}\right)^{\frac{1}{2}}.$$

From this it can be shown by straightforward algebra that

$$\frac{1}{c^2}\left(\frac{dR}{d\tau}\right)^2 = -1 + \frac{G}{c^2}\frac{(m_1 + m_2)}{R} + \frac{G^2}{c^4}\frac{(m_2 - m_1)^2}{K^2} + \frac{1}{4}\frac{K^2}{R^2}$$

where

$$K = \frac{GA}{c^3}\left[\mu_0^2 c^2 + \alpha_0^2/R^2\right]^{\frac{1}{2}}.$$

We then have the important result that in the Newtonian limit this reduces to the result obtained by Campbell and Hénon, viz,

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{Gm_1}{R} + \frac{1}{2}\frac{G\Delta m}{R} - \frac{h^2}{2R^2}$$

where $h = \dfrac{\alpha_0}{\mu_0}$ is the angular momentum per unit mass and $\Delta m = m_2 - m_1$ is the increment in the mass, to Newtonian order, due to the presence of the shell. The $\frac{1}{2}$ in front of the second term on the right hand side is a subtle term that comes about, due, in Newtonian order, to the gravitational potential energy of the shell acting on itself. The fact that the general relativity result has this precise Newtonian limit refutes Yilmaz's claim that general relativity is only a 1–body problem and gives the wrong answer for shells of matter.

For completeness the proof that the general relativity result reduces to the Campbell–Hénon formula is given.

We have in the first place that $K = \dfrac{4\pi G}{c^2} R^2 \sigma$ so

$$\begin{aligned} \frac{1}{c^2}\left(\frac{dR}{d\tau}\right)^2 &= -1 + \frac{G(m_1+m_2)}{c^2 R} + \frac{1}{4}\frac{16\pi^2 G^2 R^2 \sigma^2}{c^4} + \frac{G^2}{c^4}\frac{(m_2-m_1)^2}{16\pi^2 G^2 R^4 \sigma^2/c^2} \\ &= -1 + \left(\frac{m_2-m_1}{4\pi R^2 \sigma}\right)^2 + \frac{G(m_1+m_2)}{c^2 R} + \frac{4\pi^2 G^2 R^2 \sigma^2}{c^4}. \end{aligned}$$

We now need to compute the expression $\left(\dfrac{m_2-m_1}{4\pi R^2\sigma}\right)^2$ correct to terms in $\dfrac{1}{c^2}$. We have

$$\begin{aligned} \left(\frac{m_2-m_1}{4\pi R^2\sigma}\right)^2 &= \left(\frac{m_2-m_1}{A\mu_0}\right)^2 \left(1 + \frac{\alpha_0^2}{\mu_0^2 c^2 R^2}\right)^{-1} \\ &= \left(\frac{m_2-m_1}{A\mu_0}\right)^2 - \left(\frac{m_2-m_1}{A\mu_0}\right)^2 \frac{\alpha_0^2}{\mu_0^2 c^2 R^2}. \end{aligned}$$

Now since A is the total number of particles in the shell, we have to Newtonian order $m_2 = m_1 + A\mu_0$ so that $\left(\dfrac{m_2-m_1}{A\mu_0}\right) = 1$ and we are left with

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{1}{2}\frac{G(m_1+m_2)}{R} - \frac{h^2}{2R^2}$$

since to this order $\tau = t$. A simple rearrangement then gives the Campbell–Hénon result.

6. Conclusion

We have now investigated some aspects of Yilmaz's assertions that general relativity is essentially a 1–body test particle theory of gravitation, that it does not contain 2–body solutions, and that it does not give correct answers for the gravitational attraction between two infinite plane slabs of matter or between two concentric shells of matter.

In the first place we have seen the the solution claimed by Yilmaz for a configuration of two parallel infinite plane slabs of matter is not a solution of the Einstein field equations. In any case, the configuration described by Yilmaz (plane shells at a *fixed* distance apart) does not make physical sense, and requires unphysical stresses for its maintenance, as was pointed out inter alia by Horský long ago.

Second, we have seen that general relativity does in fact give the correct Newtonian limit in the case of both plane and spherical shells of matter. In the spherical case the Newtonian result itself, to which general relativity reduces, has a term which clearly indicates that we are dealing with a many–body theory. Consequently Yilmaz's assertion that general relativity is a 1–body theory which does not reduce to the correct Newtonian result in the case of planes shells or spherical shells must be rejected.

Furthermore, we have seen clear indications that the reason for Yilmaz's incorrect assertions is that he has imposed on general relativity extraneous ideas about the form of the metric for plane distributions of matter, and he has not computed solutions in general relativity according to the well–understood procedures which are clearly written up in the standard expositions of general relativity, e.g., the book by Misner, Thorne and Wheeler [11]. In the case of spherical shells he does not seem to have performed any calculations.

Finally it needs to be pointed out that, apart from the calculations recapitulated above, there are many more examples of careful and detailed calculations in general relativity that demonstrate that this theory is a many–body theory of gravitation which gives close agreement with observation. In particular, the beautiful work on the gravitational–radiation damping of compact binary systems which has been carried out by Blanchet, Damour, Iyer, Will and Wiseman [2] is such an example.

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