Polarisation of instantons in the SO(4) gauge theory results in gravity

M.Yu. Kuchiev*
School of Physics, University of New South Wales,
NSW 2052, Australia

Abstract

Conventional non-Abelian SO(4) gauge theory describes gravity if the gauge field possesses the specific polarised vacuum state. In this vacuum the instantons and anti-instantons have a preferred direction of orientation. Their orientation plays a role of the order parameter for the polarised phase of the gauge field. The interaction of a weak and smooth gauge field with the polarised vacuum is described by an effective long-range action which is identical to the Hilbert action of general relativity. In the classical limit this action results in the Einstein equations of general relativity. Gravitational waves appear as the mode describing propagation of gauge field which strongly interacts with the oriented instantons. The Newton gravitational constant describes the density of the considered phase of the gauge field. The radius of the instantons under consideration is comparable with the Planck radius.

1. Introduction

I wish to show that gravity arises as a particular effect in the conventional $SO(4)$ gauge theory. This paper presents in detail and develops the idea first reported in Ref.[1]. The theory under consideration is the conventional Yang-Mills gauge field theory formulated in flat Minkowski space. There is no nontrivial Riemann metric on the basic level of the theory. The Lagrangian of the theory describes gauge bosons interacting with fermions and scalars. There are no gravitons in the Lagrangian. The Newton gravitational constant does not manifest itself in the Lagrangian.

Our purpose is to consider a new phase of the gauge field. In this phase the vacuum has a nontrivial structure, that leads to a strong interaction between the vacuum and long-range fluctuations of the gauge field. As a result the low-energy degrees of freedom of the gauge field acquire quite unusual and surprising properties. Firstly, they can be adequately described by the Riemann geometry based on some Riemann metric. Secondly, an effective action describing low-energy degrees of freedom of the gauge field proves to be identical to the Hilbert action of general relativity. Thirdly, in the classical approximation

*On leave from: A. F. Ioffe Physical-Technical Institute of Russian Academy of Sciences, 194021 St.Petersburg, Russian Federation.

the metric considered satisfies the Einstein equations of general relativity. Fourthly, there appear in the theory spin-2 excitations describing some particular low-energy degrees of freedom of the gauge field interacting with the nontrivial vacuum. These remarkable properties indicate that the considered gauge field construction describes gravity.

The phase considered may be described in terms of BPST instantons [2]. An instanton is known to possess eight degrees of freedom: four of them give its position, one is its radius, and the remaining three describe its orientation. These later ones play a crucial role in our discussion. In the usual phases of gauge theory orientations of instantons are arbitrary. In this paper a phase in which instantons and anti-instantons are ordered, having a preferred direction of orientation, is considered. Thus in this phase the orientation degrees of freedom of (anti)instantons are frozen. A possible way to visualise this vacuum in terms of a simple physical analogy is to compare it with the usual ferromagnetic or antiferromagnetic phases in which spins of atoms have a preferred orientation. The (anti)instantons constituting this vacuum will be called "polarised instantons" or "a condensate of polarised instantons". The vacuum itself will be referred to as "the polarised vacuum". The basic property of the phase considered will be called Instanton-Anti-Instanton Polarisation (IAP) following Ref.[1]. The density of the condensate of polarised (anti)instantons is described by a length parameter which depends on radii and separations of the polarised (anti)instantons. This length parameter is equal to the Planck radius. The Newton gravitational constant appears in the theory as the inverse density of the condensate. Excitations of the gauge field strongly interact with the condensate if the wavelength of the excitations exceeds the Planck radius. In this case the influence of the condensate is strong, it changes the nature of the excitation. Instead of a spin-1 gauge boson there appears a new excitation, which is a spin-2 graviton. For high-energy excitations whose wavelength is smaller than the Planck radius the condensate does not play a significant role. Therefore these excitations describe the usual gauge theory fields of spin $0, 1/2$ and 1. This means that effects of gravity manifest themselves only in the region of large distances, larger than the Planck radius. For smaller distances gravity disappears. The quantisation of the theory is straightforward because basically it is a gauge theory. In particular there is no difficulty with the renormalisability because for the short-distance region the theory reveals the usual properties of gauge theory.

IAP, the basic necessary property of the vacuum in the picture considered, is introduced in Section 3. as a postulate. Certainly one question to be addressed is whether IAP can be derived in the framework of the gauge theory. A mechanism which can result in the IAP phase of the $SO(4)$ gauge theory was considered in recent Refs.[3],[4]. There gauge theory models were suggested in which there appears an interaction between instantons making their identical orientation more preferable. The possibility of the phase transition into a IAP state was verified in [4] in the framework of mean field approximation.

2. The pair of instanton and anti-instanton

For the $SO(4)$ gauge group the instantons and anti-instantons can belong to any one of the two available $su(2)$ subalgebras, $so(4) = su(2) + su(2)$. In order to focus our attention on those (anti)instantons which are useful for our purposes a specific instanton-anti-instanton pair is considered in this Section. Let us choose the generators for one $su(2)$ to be $(-1/2)\eta^a_{ij}$ and the generators for the other to be $(-1/2)\bar\eta^a_{ij}$, and refer to these algebras as $su(2)\eta$ and $su(2)\bar\eta$. Here $\eta^a_{ij}, \bar\eta^a_{ij}$ are the 't Hooft symbols [5], $a = 1,2,3,\ ij = 1,\cdots,4$. The strength of the gauge field in this notation is

$$F^{ij}_{\mu\nu} = -\frac{1}{2}(F^a_{\mu\nu}\eta^a_{ij} + \bar F^a_{\mu\nu}\bar\eta^a_{ij}) \ , \tag{1}$$

where $F^a_{\mu\nu}$ belongs to $su(2)\eta$ and $\bar F^a_{\mu\nu}$ belongs to $su(2)\bar\eta$. Consider an instanton belonging to $su(2)\bar\eta$ in the external gauge field $F^{ij}_{\mu\nu}$ in Euclidean formulation of the theory. According to [7],[6] there exists a contribution to the action describing the interaction of the instanton with the field

$$\Delta S_{\rm I} = -\frac{\pi^2\rho_{\rm I}^2}{g^2}\bar\eta^a_{\mu\nu}\bar\eta^b_{ij}\bar C^{ab}F^{ij}_{\mu\nu}(x_{\rm I}) \ . \tag{2}$$

Here $\rho_I, x_I, \bar C^{ab} \in SO(3)$ are the radius of the instanton, its position, and the matrix describing its orientation. Similarly the interaction of an anti-instanton belonging to the other subalgebra $su(2)\eta$ with the gauge field is described by the action

$$\Delta S_{\rm AI} = -\frac{\pi^2\rho_{\rm AI}^2}{g^2}\eta^a_{\mu\nu}\eta^b_{ij}C^{ab}F^{ij}_{\mu\nu}(x_{\rm AI}) \ . \tag{3}$$

Here $\rho_{AI}, x_{AI}, C^{ab} \in SO(3)$ are the radius, position, and the orientation matrix of the anti-instanton. Let us consider now a pair which consists of an instanton belonging to $su(2)\bar\eta$ and anti-instanton belonging to $su(2)\eta$. These two topological objects are in different $su(2)$ subalgebras and therefore they do not interact with each other. Suppose that their radii are equal, $\rho_I = \rho_{AI} = \rho$. Suppose also that the external field does not strongly vary with respect to the distance $x_{\rm I} - x_{\rm AI}$, $F^{ij}_{\mu\nu}(x_{\rm I}) \approx F^{ij}_{\mu\nu}(x_{\rm AI}) = F^{ij}_{\mu\nu}(x_0)$, where x_0 is the position of the pair $x_0 \approx x_{\rm I} \approx x_{\rm AI}$. Then the action describing the interaction of the instanton and anti-instanton with the external field can be found simply as a sum of the right hand sides of Eqs. (2), (3)

$$\Delta S_{\rm I,AI} = -\frac{\pi^2\rho^2}{g^2}(\eta^a_{\mu\nu}\eta^b_{ij}C^{ab} + \bar\eta^a_{\mu\nu}\bar\eta^b_{ij}\bar C^{ab})F^{ij}_{\mu\nu}(x_0) \ . \tag{4}$$

Now introduce the matrix

$$h^{ij} \in SO(4) \tag{5}$$

which rotates the generators of the gauge transformations according to

$$h^{ik}h^{jl}\eta^a_{kl} = C^{ab}\eta^b_{ij}, \quad h^{ik}h^{jl}\bar\eta^a_{kl} = \bar C^{ab}\bar\eta^b_{ij} \ . \tag{6}$$

It is clear that for any pair C^{ab}, $\bar C^{ab} \in SO(3)$ there exists $h^{ij} \in SO(4)$ satisfying Eq. (6). One can say that h^{ij} describes the orientation of the instanton-anti-instanton pair considered. Eq. (4) may be presented with the help of h^{ij} in a compact form

$$\Delta S_{\rm I,AI} = -\frac{4\pi^2\rho^2}{g^2}h^{i\mu}h^{j\nu}F^{ij}_{\mu\nu}(x_0) \ , \tag{7}$$

which will be very useful in the following discussion. Deriving Eq. (7) the identity

$$\eta^a_{\mu\nu}\eta^a_{ij} + \bar{\eta}^a_{\mu\nu}\bar{\eta}^a_{ij} = 2(\delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu}) \tag{8}$$

was used. Remember that the Latin letters i, j label the indexes of variables in the isotopic space while the Greek letters μ, ν label the indexes in the coordinate space. The symbols $\eta^a_{\mu\nu}$, $\bar{\eta}^a_{\mu\nu}$ are used to describe the orientation of instantons and anti-instantons in the coordinate space. In contrast, the symbols η^a_{ij}, $\bar{\eta}^a_{ij}$ are the generators of two $su(2)$ gauge subalgebras, see Eq. (1). According to Eq. (8) there appears a correspondence between the indexes of coordinate space and indexes of isotopic space. This fact was used writing the matrix $h^{i\mu}$ in Eq. (7) with one Latin index and one Greek one

$$h^{i\mu} = h^{ik}\delta_{k\mu} \ . \tag{9}$$

Notice that the correspondence between indexes of different spaces is a manifestation of the known property of an instanton: it is transformed identically by gauge and coordinate transformations.

It is important that the action Eq. (7) has an algebraic structure which is very close to the algebraic structure of the Lagrangian of general relativity. In the following Section a way to transform this similarity into identity is suggested.

3. The condensates of instantons and anti-instantons

Eq. (7) gives an action which depends on the gauge field at the particular point x_0 where the instanton-anti-instanton pair in question is located. Let us generalise this result considering a finite concentration of pairs which are similar to the single pair considered in the previous Section. Our first step is to consider the vicinity V of some point x_0. Let the radius r_V of this vicinity be much larger than the radius $r_{\rm mic}$ which characterises the microscopic quantum fluctuations of the gauge field considered, $r_V \gg r_{\rm mic}$. At the same time let the radius of the vicinity be small compared with the radius $r_{\rm mac}$ which characterises the variation of a gravitational field which we are going to describe, $r_V \ll r_{\rm mac}$.

Remember that our major goal is to eliminate a geometry as a basic guiding principle from the theory. It is instructive, however, to use a geometrical idea at this point of the discussion. We consider a local Galilean reference frame, a falling elevator reference frame, and assume that the microscopic physical picture in this reference frame looks simpler than in any other coordinates. It is important to emphasise that this approach is adopted in this Section in order to simplify presentation of the properties of the order parameter of the IAP phase.

Let us choose coordinates in V which give a Galilean reference frame at the point x_0. The inequality $r_V \ll r_{\rm mac}$ shows that the chosen coordinates give approximately a Galilean reference frame at every point $x \in V$. A clear way to generalise the result of the previous section is to suppose that in the vicinity of any point $x \in V$ there exists an instanton-anti-instanton pair, which is similar to the one considered in the previous section. In this pair the instanton belongs to $su(2)\bar{\eta}$ and the anti-instanton to $su(2)\eta$. In order to simplify our discussion let us imagine for a time that the dilute gas approximation is valid. Note that the physical picture considered may remain true even if the conditions of applicability of the dilute gas approximation are violated (see discussion below). Thus consider the gas of $su(2)\bar{\eta}$ instantons and $su(2)\eta$ anti-instantons. Certainly the radii ρ of (anti)instantons are supposed to be smaller than the radius of the vicinity considered, $\rho \sim r_{mic} \ll r_V$. Let us examine the interaction of this gas with the external gauge field supposing that the field strength is small, $|F^{ij,\mu\nu}| \ll 1/(g^2\rho^2)$ and varies smoothly on the instanton radius ρ. The interaction of this field with the (anti)instantons is described by the sum of the terms given in Eq. (4) resulting in the action

$$\Delta S = -\frac{\pi^2}{g^2}[\eta^a_{\mu\nu}\eta^b_{ij}\sum_k C^{ab}_k \rho_k^2 F^{ij}_{\mu\nu}(x_k) + \bar{\eta}^a_{\mu\nu}\bar{\eta}^b_{ij}\sum_l \bar{C}^{ab}_l \rho_l^2 F^{ij}_{\mu\nu}(x_l)] \ . \tag{10}$$

Here the index k enumerates the anti-instantons, l enumerates the instantons, x_k, ρ_k, $C^{ab}_k \in SO(3)$ are the position, radius and orientation matrix of k-th anti-instanton, x_l, ρ_l, $\bar{C}^{ab}_l \in SO(3)$ - the position, radius and orientation matrix of l-th instanton. The summation in Eq. (10) runs over (anti)instantons in the vicinity considered, $x_k, x_l \in V$. Our next goal is to derive the effective action describing the interaction of the external gauge field with the topological objects considered. With this purpose let us average the action Eq. (10) over the short-range fluctuations in the vacuum. The result reads

$$\Delta S = -\int [\eta^a_{\mu\nu}\eta^b_{ij}\mathcal{M}^{ab}(x) + \bar{\eta}^a_{\mu\nu}\bar{\eta}^b_{ij}\bar{\mathcal{M}}^{ab}(x)]F^{ij}_{\mu\nu}(x)d^4x \ . \tag{11}$$

Here

$$\mathcal{M}^{ab}(x) = \pi^2\langle\frac{1}{g^2}\rho^2 C^{ab}\ n(\rho, C, x)\rangle \ , \qquad \bar{\mathcal{M}}^{ab}(x) = \pi^2\langle\frac{1}{g^2}\rho^2\bar{C}^{ab}\bar{n}(\rho, \bar{C}, x)\rangle \ , \tag{12}$$

where $n(\rho, C, x)$ is the concentration of the anti-instantons having the radius ρ and orientation $C = C^{ab}$, and $\bar{n}(\rho, \bar{C}, x)$ is the concentration of instantons with the radius ρ and orientation $\bar{C} = \bar{C}^{ab}$. The brackets $\langle\ \rangle$ denote averaging over short-range fluctuations of the gauge field. For the dilute gas picture considered this should include the averaging over positions, radii and orientations of instantons. One can assume that the concentrations $n(\rho, C, x), \bar{n}(\rho, \bar{C}, x)$ depend on x provided this is a smooth dependence, negligible in the region of separation between instantons.

For the well-known phases of the gauge field – the confinement phase, the Higgs phase and the others – the probability for the instanton or anti-instanton to have some orientation does not depend on the orientation itself. Thus in these phases averaging over the orientations C^{ab}, $\bar{C}^{ab}$ in Eqs. (12) result in zero values for $\mathcal{M}^{ab}$, $\bar{\mathcal{M}}^{ab}$.

We are interested in the phase in which $su(2)\bar{\eta}$ instantons and $su(2)\eta$ anti-instantons having the preferred orientation, are polarised. This means that the probability for (anti)instanton to have some orientation depends on the orientation itself. There are preferred, more probable, orientations for them. As a result the matrices $\mathcal{M}^{ab}(x)$ and $\bar{\mathcal{M}}^{ab}(x)$ take nonzero values. Generally speaking, averaging of the orthogonal matrices

C^{ab}, $\bar{C}^{ab}$ may result in any 3×3 matrices depending on the way the instantons are ordered. Therefore, one can imagine a number of possible phases arising for different possible orderings of instantons. We are interested in the particular phase for which the averaged matrix remains orthogonal, up to some coefficient which should characterise the intensity of the condensate. The necessity of this condition is clear from the example of the instanton-anti-instanton pair considered above. Thus let us suppose that the ordering of instantons results in the following conditions

$$\mathcal{M}^{ab}(x) = \frac{1}{4}\, f\, C^{ab}(x), \qquad \bar{\mathcal{M}}^{ab}(x) = \frac{1}{4}\, \bar{f}\, \bar{C}^{ab}(x), \tag{13}$$

where $C^{ab}(x) \in SO(3)$ is the orthogonal matrix describing an orientation of the condensate of anti-instantons belonging to $su(2)\eta$, and $\bar{C}^{ab}(x) \in SO(3)$ describes an orientation of the condensate of instantons belonging to $su(2)\bar{\eta}$.

The dimensional constants f, $\bar{f}$ describe the intensity of the two condensates considered, while the coefficients $1/4$ chosen in Eq. (13) simplify the following formulas. Remember now that instantons and anti-instantons are transformed one into another by inversion. Therefore the conservation of parity depends on the properties of the condensates of instantons and anti-instantons. In order to preserve the parity conservation law the intensities of the two condensates should be equal, $f = \bar{f}$. The constant f, as seen from Eq. (12), depends on radii and separations of those instantons which belong to the condensate, as well as on the gauge coupling constant.

Now one can follow an approach similar to the one described by Eqs. (5)−−(7). Namely, let us introduce the matrix $h^{ij}(x) \in SO(4)$ which satisfies conditions similar to Eq. (6)

$$h^{ik}(x)h^{jl}(x)\eta^a_{kl} = C^{ab}(x)\eta^b_{ij}\ , \qquad h^{ik}(x)h^{jl}(x)\bar{\eta}^a_{kl} = \bar{C}^{ab}(x)\bar{\eta}^b_{ij}\ . \tag{14}$$

Then from Eqs. (11), (13) *and* (14) one finds the action

$$\Delta S = -f \int_V h^{i\mu}(x)h^{j\nu}(x)F^{ij}_{\mu\nu}(x)d^4x\ . \tag{15}$$

Integration here is restricted to the vicinity V of the point x_0 considered. In Eq. (15) a notation

$$h^{i\mu}(x) = h^{ik}(x)\delta_{k\mu}\ , \tag{16}$$

similar to Eq. (7) is used. In deriving Eq. (15) equation (8) was used.

The integrand in Eq. (15) remains invariant under two types of transformations. It is obviously invariant under gauge transformations

$$F^{ij}_{\mu\nu}(x) \to F'^{ij}_{\mu\nu}(x) = O^{ik}(x)O^{jl}(x)F^{kl}_{\mu\nu}(x)\ , \tag{17}$$

$$h^{i\mu}(x) \to h'^{i\mu}(x) = O^{ik}(x)h^{k\mu}(x)\ , \tag{18}$$

where the matrix $O^{ij}(x) \in SO(4)$ describes a gauge transformation. It is invariant as well under coordinate transformations

$$x_\mu \quad \to x'_\mu\ , \tag{19}$$

$$F^{ij}_{\mu\nu}(x) \quad \to F'^{ij}_{\mu\nu}(x') = \frac{\partial x_\lambda}{\partial x'_\mu}\frac{\partial x_\rho}{\partial x'_\nu}F^{ij}_{\lambda\rho}(x) \ , \tag{20}$$

$$h^{i\mu}(x) \quad \to h'^{i\mu}(x') = \frac{\partial x'_\mu}{\partial x_\nu}h^{i\nu}(x) \ . \tag{21}$$

Remember that the quantity $h^{i\mu}(x)$ was defined above in the Galilean reference frame in which it has a particular structure $h^{i\mu}(x) \in SO(4)$. The transformation Eq. (21) gives a definition of this matrix in arbitrary coordinates. According to definition (21) $h^{i\mu}(x)$ may not belong to $SO(4)$. In particular its determinant according to Eq. (21) is equal to the Jacobian

$$\det\left[h'^{i\mu}\right] = \det\left[\frac{\partial x'_\mu}{\partial x_\nu}\right] \ , \tag{22}$$

and may differ from unity. Rewriting Eq. (15) in arbitrary coordinates one gets

$$\Delta S = -f\int h^{i\mu}(x)h^{j\nu}(x)F^{ij}_{\mu\nu}(x)\det h(x)\, d^4x \ . \tag{23}$$

It is convenient for further discussion to define $h^i_\mu(x)$ as the inverse matrix $h^{i\mu}(x)h^j_\mu(x) = \delta_{ij}$, $h^{i\mu}(x)h^i_\nu(x) = \delta_{\mu\nu}$. The quantity $\det h(x)$ in Eq. (23) is the determinant of this matrix $\det h(x) = \det[h^i_\mu(x)] = (\det[h^{i\mu}(x)])^{-1}$, which according to Eq. (22) describes the Jacobian of the coordinate transformation.

Up to this point our discussion was restricted to the small neighborhood V of the point x_0 where one is able to choose Galilean coordinates to begin with. The final expression found Eq. (23) has a general form valid in any coordinate frame. This fact permits an easy extension of this result. Really, one can choose now any point in space. We assume that IAP takes place. It means that in the Galilean coordinates in the vicinity of this point there are polarised instantons $\in su(2)\bar\eta$ and polarised anti-instantons $\in su(2)\eta$. Their interaction with the external gauge field is described by Eq. (23), if the integration in this formula is restricted to the vicinity of the point considered. Summing contributions of the vicinities of different points in space one finds that Eq. (23) may be applied to all the space described in arbitrary coordinates.

We will call the vacuum satisfying Eqs. (12), (13) the *IAP phase*. The matrices $C^{ab}(x)$, $\bar C^{ab}(x)$ describing the orientations of anti-instantons and instantons play the role of the order parameter for this phase. This order parameter may be thought of as the $C^{ab}(x)\times\bar C^{ab}(x) \in SO(3)\times SO(3)$ matrix. The equivalent definition of the order parameter is given by the matrix $h^{i\mu}(x) \in SO(4)$ defined in Eq. (14). It describes orientation of both instantons and anti-instantons. It is important that these definitions of the order parameter are valid only in the Galilean reference frame. The transformation of the order parameter to arbitrary coordinates given by Eq. (21) results in the fact that in arbitrary coordinates the order parameter satisfies a condition

$$\frac{h^{i\mu}(x)}{[\det h(x)]^{1/4}} \in SL(4) \ . \tag{24}$$

Eq. (23) plays a very important role in the following discussion. In deriving it we used Galilean coordinates, thus giving reference to the geometry based idea. This was done to simplify the presentation. The approach considered does not rely on the geometry of the space-time. Therefore it is important to formulate the idea without any reference to the geometry. In order to do this let us keep in mind that we suppose space-time to be basically flat. Therefore there are initial basic coordinates of this flat space-time in which the gauge theory is formulated. Let us suppose that in these coordinates all four possible topological excitations in the vacuum are polarised, i.e., there are polarised instantons and anti-instantons in both $su(2)\eta$ and $su(2)\bar{\eta}$ subalgebras. The orientation of the $k-th$ topological object may be described with the help of 6×6 matrix S_k

$$S_k = \begin{pmatrix} C_k & D_k \\ \bar{D}_k & \bar{C}_k \end{pmatrix} . \tag{25}$$

Here $C_k, \bar{C}_k, D_k, \bar{D}_k \in SO(3)$. C_k describes the orientation if the k-th topological object is the anti-instanton $\in so(2)\eta$ gauge subalgebra, D_k describes the orientation if it is the instanton $\in so(2)\eta$, $\bar{D}_k$ - if it is the anti-instanton $\in so(2)\bar{\eta}$, and $\bar{C}_k$ - if it is the instanton $\in so(2)\bar{\eta}$. Following steps very similar to those in the discussion above, one can show that the interaction of (anti)instantons with the external gauge field is described by the action

$$\Delta S = -\int \eta^A_{\mu\nu}\eta^B_{ij}\mathcal{M}^{AB}(x)F^{ij}_{\mu\nu}(x)d^4x \ , \tag{26}$$

where the 6×6 matrix $\mathcal{M}^{AB}$, $A, B = 1,\cdots,6$ describes the averaged orientation of all topological objects available in the vacuum

$$\mathcal{M}^{AB}(x) = \pi^2\langle \frac{1}{g^2}\rho^2 S^{AB} n(\rho, S, x)\rangle \ . \tag{27}$$

Here $n(\rho, S, x)$ is the concentration of (anti)instantons having the radius ρ, and the orientation described by the matrix S defined in Eq. (25).

In order to reproduce Eq. (23) this orientation should satisfy particular conditions. In particular, it should have the following form

$$\mathcal{M}^{AB}(x) = \frac{1}{4} f\, M^{AB}(x) \ , \tag{28}$$

where f is a constant characterising the density of the condensate (compare Eq. (13) above), and the 6×6 matrix $M^{AB}(x)$ satisfies

$$M^{AB}(x) \in SO_+(3,3) \ . \tag{29}$$

Remember that $M \in SO(3,3)$ means that

$$M\Sigma M^T = \Sigma \ , \qquad \Sigma = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix} . \tag{30}$$

where numbers with hats represent 3×3 diagonal matrices. The notation $SO_+(3,3)$ is used to describe the subset of all those matrices M, $M \in SO(3,3)$, which can be transformed into the unity matrix by a continuum transformation in the $SO(3,3)$ group.

In this approach the matrix $M^{AB}(x) \in SO_+(3,3)$ plays the role of an order parameter. Remember now that there is the known homomorphism between matrices belonging to $SL(4)$ and matrices belonging to $SO(3,3)$, $SL(4) \cong SO(3,3)$ [8]. This homomorphism may be presented as a statement that for any $M^{AB}(x) \in SO_+(3,3)$ there exists a matrix $h^{i\mu}(x) \in SL(4)$ satisfying an equality

$$\eta^A_{\mu\nu}\eta^B_{ij}M^{AB}(x) = 2\left(h^{i\mu}(x)h^{j\nu}(x) - h^{i\nu}(x)h^{j\mu}(x)\right) \ . \tag{31}$$

Here generalised 't Hooft symbols are introduced $\eta^A = \eta^a$ if $A = a = 1,2,3$ and $\eta^A = \bar\eta^a$ if $A - 3 = a = 1,2,3$. Eq. (31) shows that the order parameter may be considered not only as a $M^{AB}(x) \in SO(3,3)$ matrix, but as $h^{i\mu}(x) \in SL(4)$ matrix as well. Remember that we use the basic, initial coordinates.

Substituting Eq. (31) into Eqs. (26), (28) one finds that the interaction of the external gauge field with the vacuum in the basic coordinates is described by the action Eq. (15). Using the transformation to the arbitrary coordinates Eq. (21) we derive the desired Eq. (23).

We come to the following definition of polarisation: IAP means that all four topological excitations available in the $SO(4)$ gauge group are polarised in the initial basic coordinates. Their polarisation defined in Eqs. (25), (27), (28) should satisfy Eq. (29). The matrix $M^{AB}(x)$ plays the role of the order parameter for the IAP phase. The matrix $h^{i\mu}(x)$ satisfying Eq. (31) gives an alternative possibility for describing the order parameter. The physical meaning of Eq. (29) is simple. It states that there always exist particular local coordinates in which there are only two polarised gases, one of them is the gas of polarised instantons $\in su(2)\bar\eta$, and the other one is the gas of polarised anti-instantons $\in su(2)\eta$.

The most important property of the IAP phase is the fact that the gauge field interacts with the vacuum. This interaction is described by the action Eq. (23).

4. The Einstein equations

The action Eq. (23) describes the interaction of the order parameter in the IAP phase with a gauge field. Consider the classical approximation. The field $F^{ij}_{\mu\nu}(x)$ is supposed to be weak and smooth. This means that it has the trivial topological structure on the microscopic level. In contrast, the order parameter $h^{i\mu}(x)$ describes those degrees of freedom of the gauge field which are associated with instantons and therefore have highly nontrivial microscopic topological structure. Thus $F^{ij}_{\mu\nu}(x)$ and $h^{i\mu}(x)$ describe the states of the gauge field with quite different topological structure. Different topology enables one to use them as a set of two independent variables. Denoting the vector potential of the external field $F^{ij}_{\mu\nu}(x)$ by $A^{ij}_\mu(x)$ one can consider the action Eq. (23) as the functional $\Delta S = \Delta S(h^{i\mu}(x), A^{ij}_\mu(x))$. Then the classical equations read

$$\frac{\delta(\Delta S)}{\delta A^{ij}_\mu(x)} = 0 \ , \tag{32}$$

$$\frac{\delta(\Delta S)}{\delta h^{i\mu}(x)} \;=\; 0\ . \tag{33}$$

From Eq. (32) one finds

$$\nabla^{ik}_{\mu}[\big(h^{k\mu}(x)h^{j\nu}(x) - h^{k\nu}(x)h^{j\mu}(x)\big)\ \det h(x)] = 0\ . \tag{34}$$

Here $\nabla^{ij}_{\mu} = \partial_\mu \delta_{ij} + A^{ij}_{\mu}(x)$ is the covariant derivative in the gauge field. Eq. (33) gives

$$h^{j\nu}(x)F^{ij}_{\mu\nu}(x) - \frac{1}{2}h^{i}_{\mu}(x)h^{k\lambda}(x)h^{j\nu}(x)F^{kj}_{\lambda\nu}(x) = 0\ . \tag{35}$$

In order to present Eqs. (34), (35) in a more convenient form let us define three quantities, $g_{\mu\nu}(x), \Gamma^{\lambda}_{\mu\nu}(x)$, and $R^{\lambda}_{\rho\mu\nu}(x)$:

$$\begin{aligned} g_{\mu\nu}(x) &= h^{i}_{\mu}(x)h^{i}_{\nu}(x)\ , & (36)\\ \Gamma^{\lambda}_{\mu\nu}(x) &= h^{i\lambda}(x)h^{j}_{\mu}(x)A^{ij}_{\nu}(x) + h^{i\lambda}(x)\partial_\nu h^{i}_{\mu}(x)\ , & (37)\\ R^{\lambda}_{\rho\mu\nu}(x) &= h^{i\lambda}(x)h^{j}_{\rho}(x)F^{ij}_{\mu\nu}(x)\ . & (38) \end{aligned}$$

Remember that space-time under consideration is basically flat. Therefore Eqs. (36), (37), (38) just define the left-hand sides. From (36), (37) one finds that Eq. (34) may be presented in the form $\Gamma^{\lambda}_{\mu\nu} = (1/2)g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$. It demonstrates that we may consider $g_{\mu\nu}(x)$ as a metric and $\Gamma^{\lambda}_{\mu\nu}(x)$ as a Christoffel symbol. Moreover, one finds that the quantity $R^{\lambda}_{\rho\mu\nu}(x)$ introduced in Eq. (38) turns out to be equal to the Riemann tensor. Considering now the second classical equation (35) one verifies with the help of Eqs. (36), (38) that it results in the Einstein equations of general relativity in the absence of matter

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\ . \tag{39}$$

Remember that up to this point our discussion took place in Euclidean space. Our final result, Eq. (39) may be transformed into Minkowski space. We come to the important conclusion. If IAP takes place in the $SO(4)$ gauge theory then the classical approximation for this gauge theory results in gravity. It means that there appears a Riemann metric for which the Einstein equations are valid. These equations imply, in particular, that there exist gravitational waves. That is an important result since the initial gauge theory possesses no graviton on the basic level. A graviton in the picture discussed appears as a coherent state of the gauge field interacting with the condensate of instantons and anti-instantons.

Consider the action (23) when the classical Eq. (32) is valid. It is clear from (36), (37), (38) that the form of the action Eq. (23) is identical to the action of general relativity S_{GR}, if the action S_{GR} is continued into Euclidean space. Note in particular that the sign of action Eq. (23) agrees with the sign of S_{GR}. One can consider them as identical quantities if the Newton gravitational constant k is identified as

$$k = \frac{1}{16\pi f}\ . \tag{40}$$

Thus the density of the condensate f provides the dimensional parameter necessary in the theory of gravity. This relation shows in particular that the radii and separations of (anti)instantons which give the contribution to the constant f, see Eqs. (12), are comparable to the Planck radius. This estimation is valid up to the factor equal to the gauge coupling constant g.

It is clear that the Riemann structure discussed gives an adequate description only if the distances considered are much larger than the Planck radius. Really, the wavelength of the graviton in the picture considered must exceed the typical separation between polarised instantons as well as their radii. For shorter wavelengths it is impossible to divide the gauge field into the short-range part described by the polarised instantons and the long-range part described by the weak external field interacting with the instantons. Thus for wavelengths smaller than the Planck radius the gauge theory describes the usual excitations, gauge bosons interacting with fermions and scalars.

5. Conclusion

In this paper we have discussed a new point of view on gravity. We postulated the existence of a particular nontrivial phase, called Instanton-Anti-Instanton Polarisation, in the vacuum state of the $SO(4)$ gauge theory. This phase appears due to polarisation of instantons and anti-instantons and is characterised by the $SO_+(3,3)$ order parameter describing orientations and relative intensities of the polarised condensates of (anti)instantons. This postulate results in a variety of very promising consequences. A Riemann metric describing the low-energy degrees of freedom of the gauge field arises, and the effective action for these degrees of freedom turns out to be identical to the Hilbert action of general relativity, which in the classical limit results in the Einstein equations. Thus the dynamics of gravity is shown to arise directly from the dynamics of gauge theory. In this sense gravity is shown to be one of the effects in gauge theory, not an independent basic theory.

It is important that gravity manifests itself only for energies below the Planck energy. For high energy excitations the condensates considered do not play a role and, therefore, they are described by the usual gauge bosons, which can interact with spinor and scalar matter fields. This means that there is no problem whatsoever with the quantisation and renormalisability of the theory, because for short distances it reduces to the usual gauge theory.

Refusing to consider geometry as a cornerstone of the theory certainly poses further challenges. For example, the equivalence principle does not follow directly from the initial assumptions in the picture we have discussed. It is necessary to find an explanation for it based on the dynamics of the gauge field which supplies us with the metric considered.

I am thankful to V.V. Flambaum, C.J. Hamer, I.B. Khriplovich, and O.P. Sushkov for critical comments, and L.S. Kuchieva for help in preparation of the manuscript. The financial support of Australian Research Council is acknowledged.

References

[1] M.Yu. Kuchiev, Europhys. Lett. **28**, 539 (1994).

[2] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin, Phys. Lett. **59B**, 85 (1975).

[3] M.Yu. Kuchiev, Europhys. Lett. **30**, 255 (1995).

[4] M.Yu. Kuchiev, Phys. Rev. D **53**, 6946 (1996).

[5] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).

[6] A.I. Vainshtein, V.I. Zakharov, V.A. Novikov and M.A. Shifman, Usp. Fiz. Nauk **136**, 553 (1982).

[7] C. Callan, K. Dashen, and D. Gross, Phys. Rev. D **17**, 2717 (1978).

[8] R. Gilmore, *Lie Groups, Lie Algebras, and some of their applications*, (John Willey & Sons, New-York, London, Sydney, Toronto 1974).