The cosmological singularity

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Abstract

Ever since the Hawking-Penrose singularity theorems of the 1960s, it has become clear that our universe possesses an unavoidable singularity in the past. The only ways to avoid the singularity using purely classical physics is either to postulate special kinds of matter or move to higher order Lagrangian theories. Alternatively one can appeal to quantum gravity, thermodynamic arguments or the discretisation of space-time to avoid the singularity predicament. None of these approaches provide totally satisfactory answers to the singularity problem.

If one stays within classical general relativity and accepts singularities as part of physics then it is necessary to define a boundary structure to space-time in a mathematically consistent and rigorous manner. Such boundary structures are briefly described here and the behaviour of cosmological models near the boundary is discussed. Particular reference is made to the blue-shift problem and the Weyl curvature hypothesis.

1. Introduction

The Penrose-Hawking incompleteness theorems of the 1960's [1] show that all space-times having certain minimally acceptable physical attributes develop singularities. If the incompleteness lies to the past of some event it is called a *cosmological singularity.* A typical example of such a theorem which resulting in a cosmological singularity is the following:

Theorem (Hawking [2]): *Let (M, g) be a space-time for which*

(1) The strong energy condition holds, i.e.

$$R_{\mu\nu}u^\mu u^\nu = T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T \geq 0$$

for all unit timelike vectors $u^\mu u_\mu = -1$.
(2) Strong causality holds.
(3) There exists a point p such that all past directed timelike geodesics through p start reconverging in a compact region to the past of p.

Then the space-time is not singularity-free (has incomplete causal curves).

The physical content of these is assumptions is as follows: (1) asserts that if ρ is the energy density and P_i are the principal pressures, then

$$\rho + P_i \geq 0 \quad \text{for } i = 1, 2, 3, \qquad \text{and} \qquad \rho + \sum_{i=1}^{3} P_i \geq 0.$$

(2) essentially requires that there be no closed timelike curves (the condition is actually slightly weaker than this), and
(3) implies that there is a closed trapped surface in the past null cone of p.

Hawking and Ellis subsequently showed [3] that the observed isotropy of the cosmic black body radiation implies that condition (3) is met, and furthermore that every event on the surface of last scatter ($z = 1000$) has a singularity in its past.

Theorems such as this demonstrate that general relativity predicts its own limitations. Our universe would appear to have an unavoidable singularity in its past, a region or boundary of which physics cannot speak. To meet this impasse two basic strategies have arisen. Either one attempts to find ways in which the conclusions of the singularity theorems can be avoided, or one accepts the conclusion and tries to make the best physical and mathematical sense one can out of a singular space-time. In this paper we summarise the main arguments and conclusions for these two strategies.

2. Avoiding the singularity

2.1 Scalar fields

While no physically sensible perfect fluid violates the strong energy condition, this is not generally true of classical field theories. For example, a scalar field ϕ subject to a potential $V(\phi) > 0$ will have density and pressure in a Robertson-Walker cosmology given by

$$\rho = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P = \tfrac{1}{2}\dot{\phi}^2 - V(\phi).$$

The strong energy condition is seen to be violated if $\dot{\phi}^2 < V(\phi)$. Since initial conditions ϕ_0 and $\dot{\phi}_0$ may be set arbitrarily, it is clearly possible to achieve this at any initial time t_0.

Consider, for example, a spatially flat Robertson-Walker metric expressed

$$ds^2 = -dt^2 + e^{2\alpha}(dx^2 + dy^2 + dz^2).$$

Adopting units such that Einstein's gravitational constant is unity, the field equations are

$$\ddot{\alpha} = -\tfrac{1}{2}\dot{\phi}^2 = -3\dot{\alpha}^2 + V(\phi).$$

Now the function $\alpha(t)$ can be arbitrarily chosen subject to the restraint $\ddot{\alpha} < 0$, we integrate the first equation to find $\phi = \phi(t)$ and read off $V(t)$. Inverting the ϕ evolution to find $t(\phi)$ we see that there always exists a potential $V(\phi)$ which has the initially postulated $\alpha(t)$ as a solution of these equations. As a trivial example let us pick $\alpha = -t^2$. This is a non-singular cosmology, which expands from an infinitely small space in the infinite past and then recontracts back in the infinite future. The scalar potential which arises from the above procedure is

$$V(\phi) = -2 + 3\phi^2.$$

This is in fact the potential of a massive scalar field, shifted by a negative constant (i.e. cosmological constant), not a totally unreasonable potential for a physicist to contemplate. While the negative cosmological constant might be objected to, it is not hard to find examples in which $V(\phi)$ is positive everywhere (a similar example has been found by Madsen [4]).

Examples such as this would however appear to be rather special, and they are probably unstable. A general theorem recently proved by my student S. Foster states that for any positive potential $V(\phi)$ which grows no faster than an exponential function as $|\phi| \to \infty$ (e.g. any polynomial potential), the generic behaviour of cosmological solutions is asymptotic to the solution corresponding to a zero mass potential $V(\phi) = 0$ [5]. It seems then that the generic behaviour of all Robertson-Walker scalar field potentials will have scaling factor $R(t) \approx t^{1/3}$ and should have a genuine singularity in the past.

2.2 Higher order Lagrangians

While alternative gravitational field theories such as Brans-Dicke can be expected to suffer a similar fate to the scalar fields discussed above, it is sometimes proposed that Lagrangians made up of higher order invariants of the curvature tensor may govern the gravitational field equations. For example, as a result of Lanczos' identity [6] the most general action up to quadratic terms in the curvature tensor has the form

$$\delta \int \sqrt{-g}\{\gamma(R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\} + L_{\text{matter}}\, d^4x,$$

which leads to fourth order field equations for $g_{\mu\nu}$.

While undoubtedly there are singularity free cosmologies in such theories the freedom available in initial data seems to be overgenerous as it is necessary to specify not only the metric components g_{ab} and their time derivatives $\dot{g}_{ab}$ on any spacelike surface $t =$ const, but their second and third time derivatives as well. The amount of freedom now available makes it very easy to circumvent the singularity theorems, but only at a cost of overspecifying initial data in a manner which most physicists would consider to be unnatural.

2.3 Quantum gravity

Probably the most frequently expressed hope is that a proper quantum theory of gravity will avoid the problem of singularities. Quantum field theory of course has singularity problems of its own, making it necessary to invent all kinds of renormalisation techniques. It seems a fairly forlorn hope that the singularities of QFT should in some magical manner cancel out the singularities arising in general relativity, and this is almost certainly not what is expected to happen by even the most ardent proponents of this view.

In the absence of a satisfactory quantum theory of gravity, the best we may have to work with at present is a heuristic such as is represented by the Wheeler-deWitt equation [7, 8]. This is essentially a kind of Schrödinger equation for the wave function of the universe Ψ, defined on *superspace*, the space of all possible 3-geometries h_{ij} in the canonical formulation of general relativity. The infinite number of degrees of freedom inherent in the Wheeler-DeWitt equation is in practice quite intractable, so one usually restricts the metrics to those possessing a high order of symmetry such as characterised by one of the Bianchi types. This has the effect of reducing superspace to a space of finitely many degrees of freedom known as *minisuperspace* and by this technique the Wheeler-DeWitt equation becomes manageable.

While the possibility exists here for avoiding the singularity at $R = 0$, particularly if one invokes the Hartle-Hawking "no-boundary" condition [9], there is considerable ambiguity in the choice of wave equation and the sorts of allowable boundary conditions. Probably even more significant is the fact that it is hard to be sure what it all can mean, as the time parameter in the Wheeler-deWitt equation is certainly no ordinary time as understood in general relativity. The evolution of the wave function of the universe seems to be happening in some kind of "super-time" of which no ready physical interpretation is available.

2.4 *Thermodynamic arguments*

A totally different line of attack on the problem of the cosmological singularity has been suggested by Bekenstein, who argues that the singularity *must* be avoided on thermodynamic grounds alone [10]. Bekenstein points out that the entropy S of any complete physical system with total energy E, enclosed within a sphere of radius R, has an upper bound given by

$$\frac{S}{E} \leq \frac{2\pi R}{\hbar c}$$

(units are such that Boltzmann's constant $k = 1$, so that entropy is dimensionless). Applying this to the entire universe, setting the radius to be the event horizon $R_H = 2ct$ (assuming pure radiation), we have at early times

$$\rho = \mathcal{N} a T^4, \qquad s = \frac{4}{3} \mathcal{N} a T^3$$

whence

$$\frac{S}{E} = \frac{s}{\rho} = \frac{4}{3T} \leq \frac{2\pi R_H}{\hbar c} = \frac{4\pi}{\hbar} t.$$

Now applying the very robust formula for temperature evolution

$$T \approx \left(\frac{45}{32} \frac{c^5 \hbar^3}{\mathcal{N} G \pi^3 t^2} \right)^{1/4},$$

we see that

$$t \gtrsim \mathcal{N}^{1/2}\sqrt{\frac{G\hbar}{c^5}} = \mathcal{N}^{1/2}t_{\text{Pl}}$$

where t_{Pl} is the *Planck time*. It would seem therefore that any time earlier than $\mathcal{N}^{1/2}t_{\text{Pl}}$ is thermodynamically impossible. Bekenstein's argument and its further extension to more general cosmologies by Schiffer [11] is rather more sophisticated than this, but all the essentials of the argument are encapsulated in the above.

However while it may be of some comfort to know that thermodynamics rules out eras close to the Planck time and therefore avoids the initial singularity, Bekenstein's argument leaves us with no alternatives at these early times. It still begs the question, what exactly does happen to the universe near the Planck time?

2.5 Discrete space-time

Another possibility is that, in the ultimate analysis, the continuum manifold picture of space-time does not hold true and some more discrete structure takes over at small enough scales [12, 13]. To see that discreteness might be expected to apply at similar scales to those arising from Bekenstein's thermodynamic argument, consider the following argument.

From the constants of classical gravitation (G and c) one can construct a unit of power

$$P^* = \frac{c^5}{G} = 3.6 \times 10^{52} \text{ joules/sec.}$$

This is the sort of power one expects to be generated in the final stages of gravitational collapse into a black hole. By the "no-hair theorems" no process could be more efficient than this and P^* must represent an upper bound to the power generation of any physical process whatsoever.

To locate a particle within a time interval Δt one needs, by the uncertainty principle, an energy $E > \hbar/\Delta t$. This involves a power generation $E/\Delta t$ and if this is to be less than the maximum power P^* we see at once that

$$\Delta t > \sqrt{\frac{G\hbar}{c^5}} = t_{\text{Pl}}.$$

Similarly the minimum locatable spatial distance for a particle would be

$$\Delta x = c\Delta t > \sqrt{\frac{G\hbar}{c^3}} = \ell_{\text{Pl}}.$$

No physical meaning can therefore be attributed to distances and times less than those defined by the Planck scale, and some kind of discrete structure must surely apply at this scale .

Bekenstein's scale, which is probably some tens of Planck scales, represents the beginning of breakdown of the classical regime. In the totally discretised pre-Planck space-time, the causal relations will have totally broken down as particle paths disintegrate into a “space-time rubble” (see figure 1). Time’s arrow arises with the growth of thermodynamics itself, which cannot occur until the Bekenstein era. While the details still have to be worked out, it is interesting to note that D. Meyer [14] has shown how the arrow of time can arise as a phase transition phenomenon in a causal lattice with an Ising-type action on it.

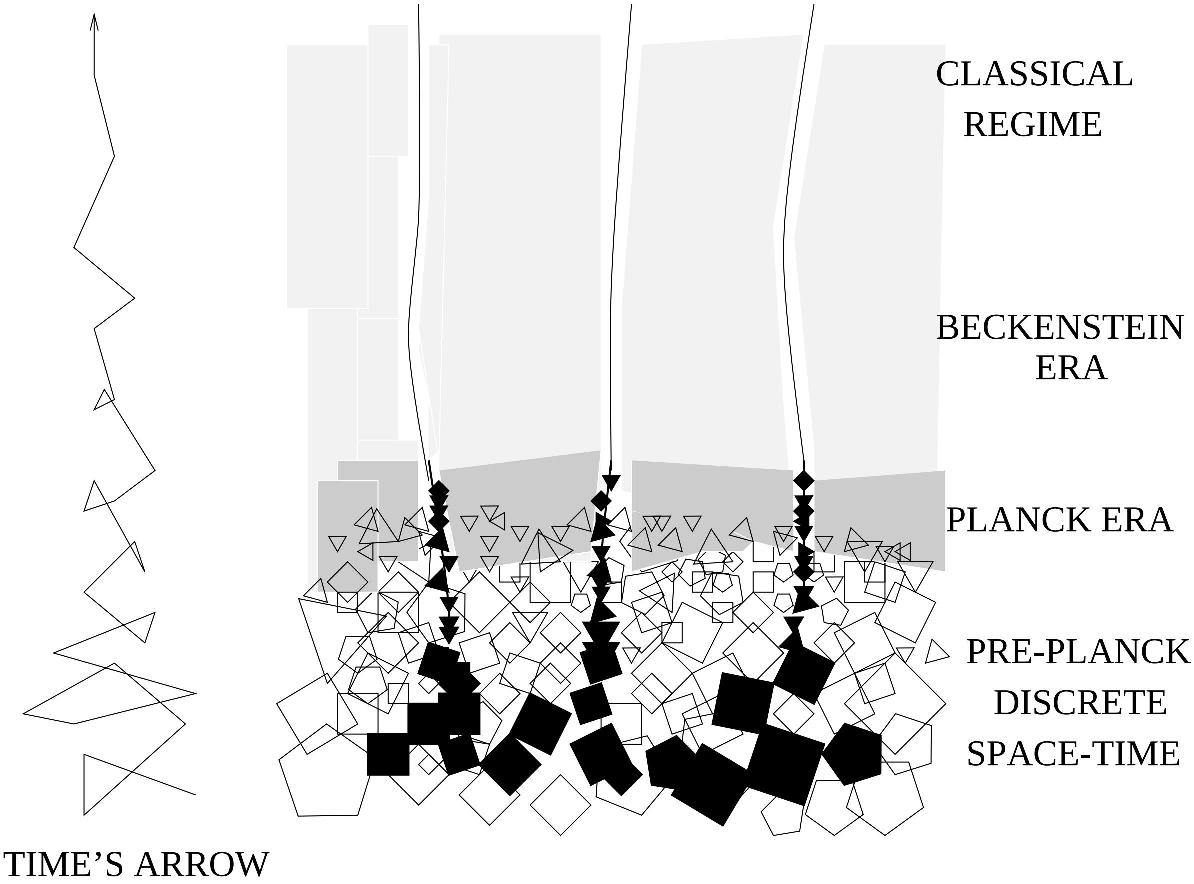

Figure 1: Transition from discrete to continuum space-time.

3. Singularity acceptance

3.1 Boundary constructions

Is it possible to stay within general relativity and somehow accept the existence of singularities as part of the theory? One of the first tasks would then be to give singularities some kind of mathematical legitimacy by defining a boundary structure for a space-time manifold.

Historically the best known definitions of a boundary for a space-time have been (a) Geroch’s *g-boundary* [15], (b) the *c-boundary* of Geroch, Kronheimer and Penrose [16], and (c) Schmidt’s *b-boundary* [17] based on the bundle of frames $L(M)$. The latter has been frequently adopted as the best available since it is applicable to much more general manifolds. Indeed, only an affine connection is needed on the base manifold to define the b-boundary. However, despite the mathematical elegance of this construction, it is prohibitively difficult to carry out in practice, and can lead to strange identifications of boundary points in even the simplest situations.

Recently Susan Scott and I gave a new boundary definition which seemed to have all the advantages of generality inherent in the b-boundary construction, yet was reasonably straightforward to apply on specific metrics [18, 19]. Our *abstract boundary* or *a-boundary* was devised in order to make rigorous the most common coordinate-minded ways of discussing singularities. In the usual situation one is presented with a coordinate patch U (an open set) at the boundary of which some metric components or their derivatives become singular. The question arises: is this singularity a real one or can an extension of the manifold be found such that the metric components become regular on the boundary ∂U?

If an extension is possible then there is an open embedding ϕ of M into a larger manifold $(\widehat{M}, \widehat{g})$ of the same dimension such that $\phi(U)$ has boundary points which are regular points of $(\widehat{M}, \widehat{g})$. The a-boundary concept attempts to display singular boundary points in a similar way, but as being "failed" boundary points of M, i.e. boundary points which cannot be regularised by *any* enveloping manifold $(\widehat{M}, \widehat{g})$.

Open embeddings $\phi : M \to \widehat{M}$ are the key to the construction and we call them *envelopments*. While the details of the abstract boundary construction are fairly involved, the essence of the definition can be visualised as follows. Consider two envelopments $\phi : M \to \widehat{M}$ and $\phi' : M \to \widehat{M}\,'$, and let p be a boundary point of the image set $\phi(M)$ by the first envelopment, and p' be a boundary point of the second envelopment. We will say the the boundary point p' *covers* p if whenever a sequence of points $x_1, x_2, \ldots$ in M approaches p then it also approaches p' and vice versa. In mathematical notation

$$p' \text{ covers } p \qquad \text{if and only if} \qquad \phi(x_n) \to p \implies \phi'(x_n) \to p' \ \forall \{x_n\} \in M.$$

p and p' are said to be *equivalent* boundary points if p' covers p and vice versa (see figure 2). The *abstract boundary* is then defined as the set of equivalence classes $[p]$ (*abstract boundary points*) defined by this equivalence relation.

A boundary point p of an envelopment $\phi : M \to \widehat{M}$ is called an *(essential) singularity* if it can be approached by a geodesic (or other specified type of curve such as a curve of bounded acceleration) with finite parameter but cannot be covered by a non-singular boundary set. The latter means a set of boundary points p through which it is possible to extend the metric g to a metric $\overline{g}$ in a neighbourhood of p in $\widehat{M}$. The notion of being an essential singularity can be shown to pass to the abstract boundary, and provides us with the concept of an *abstract singularity.*

Now let p be a boundary point of some envelopment of a space-time (M, g). We define its future set $I^+(p)$ to be the set of all points $q \in Ma$ such that there is a past directed curve from q which approaches p. Similarly one defines the past $I^-(p)$a. This notion readily passes to the abstract boundary $\mathcal{B}(M)$, so that we have a definition of the future and past of an abstract boundary point, denoted $I^\pm[p]$. A *cosmological singularity* is an abstract boundary point $[p]$ which has a future but no past, i.e. $I^+[p] \neq \emptyset$ and $I^-[p] = \emptyset$.

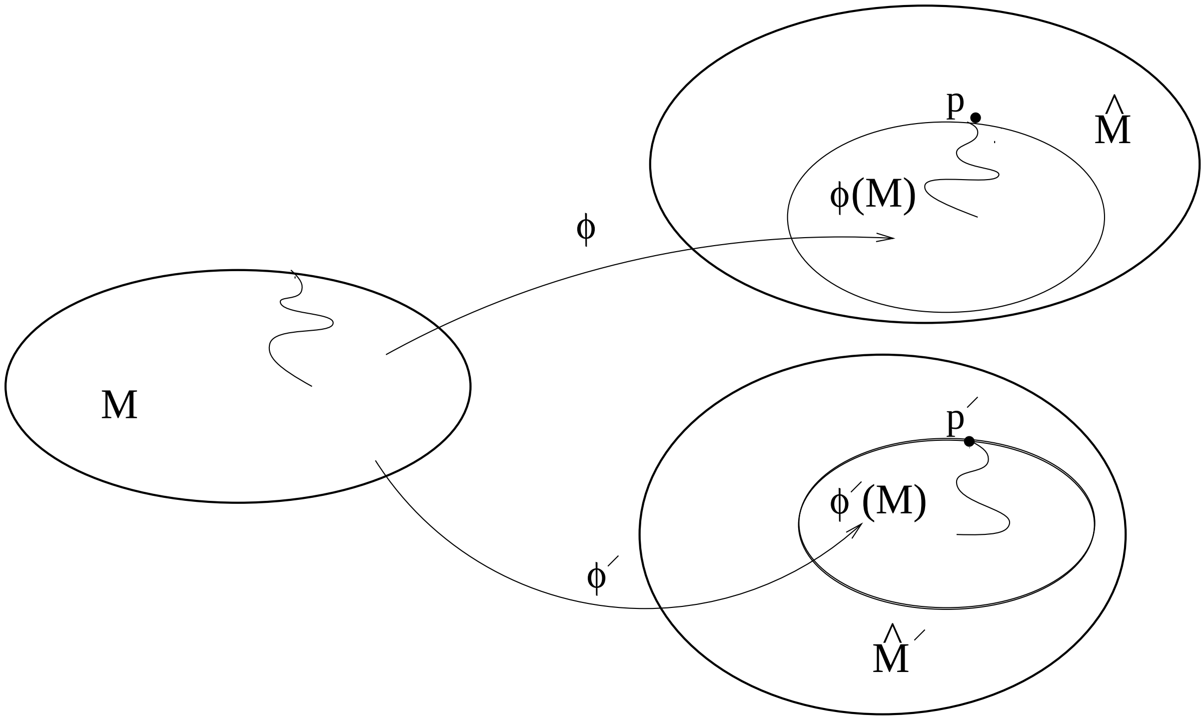

Figure 2: Equivalent boundary points

3.2 Dynamical conditions near the singularity

Now that we have a reasonably satisfactory mathematical definition of what a cosmological singularity actually is, the most important questions relate to how the space-time can be expected to behave as we approach the singularity.

In a Gaussian normal coordinate system

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j$$

the Einstein field equations read

$$\begin{aligned}
G_{00} &= \tfrac{1}{2}(R^{(3)} + K^2 - K_{ij}K^{ij}) = \kappa T_{00} \\
G_{0i} &= K_i{}^j{}_{|j} - K_{,i} = \kappa T_{0i} \\
G^i{}_j &= \dot{K}^i{}_j + K K^i{}_j + G^{(3)i}{}_j - \delta^i{}_j(\dot{K} + \tfrac{1}{2}K^2 + \tfrac{1}{2}K_{kl}K^{kl}) = \kappa T^i{}_j
\end{aligned}$$

where

$$K_{ij} = \tfrac{1}{2}\dot{g}_{ij}, \qquad K = K_{ij}g^{ij} = \frac{1}{\sqrt{g}}(\sqrt{g})^{\cdot},$$

and $\kappa = 8\pi G/c^4$.

Taking the trace of the $G^i{}_j$ equation and combining with the G_{00} equation gives the well-known *Raychaudhuri equation* [20]

$$\dot{K} + K_{ij}K^{ij} + \tfrac{1}{2}\kappa(\rho + 3\overline{P}) = 0. \tag{1}$$

This equation is of great importance for singularity theory because combining the strong energy condition with the fact that we always have the inequality $K_{ij}K^{ij} > \frac{1}{3}K^2$ immediately gives us that the metric has a singularity where $\sqrt{g} \to 0$. While in vacuum it is possible that this singularity is of a purely coordinate kind (for example, where geodesics of the Gaussian congruence start to cross each other), for simple matter models such as irrotational dust where $\rho \propto (\sqrt{g})^{-1}$ the singularity must be essential since a curvature invariant become infinite there.

Vacuum dominated singularities

In a famous series of papers by Belinskii, Khalatnikov and Lifschitz [21, 22] it was postulated that near the cosmological singularity one may reasonably expect the metric to have the form

$$g_{ij} \approx t^{2p_1}\ell_i\ell_j + t^{2p_2}m_im_j + t^{2p_3}n_in_j$$

where ℓ_i, m_i, n_i and p_i are functions of the spatial coordinates x^1, x^2, x^3, and the exponents p_i satisfy the *Kasner* conditions

$$p_1 + p_2 + p_3 = {p_1}^2 + {p_2}^2 + {p_3}^2 = 1. \tag{2}$$

This behaviour is found on setting $T^\mu_\nu \approx 0$ and $G^{(3)\mu}_{\ \ \ \nu} \approx 0$ which, if exactly true, would result in the Kasner Bianchi type I vacuum solutions. The condition $T^\mu_\nu \approx 0$ essentially postulates that near the singularity the metric should be *vacuum dominated*, i.e. the dynamics should be dominated by the ambient gravitational fields rather than the matter. On the other hand the second condition $G^{(3)\mu}_{\ \ \ \nu} \approx 0$ says that the metric should be *velocity dominated*, a condition which certainly holds true in FRW models where curvature effects become negligibly small as we approach the singularity.

Nobody knows how general these assumptions are. The papers in this series also leave some serious mathematical questions regarding their rigour. One particular oversimplification seems to be the assumption that the singularity occurs at $t = 0$, i.e. that the initial singularity occurs at the same time for all particles of the cosmic fluid. A detailed criticism of these papers has been given by Barrow and Tipler [23].

Velocity-dominated singularities

The case of velocity-dominated singularities (without any specific regard to the vacuum-dominated condition) was discussed by Eardley, Liang and Sachs for the particular case of irrotational perfect fluid [24, 25]. In the case of irrotational dust (zero pressure) they found that

$$g_{ij} \approx (t - t_0(\mathbf{x}))^{p_i}(t - t_1(\mathbf{x}))^{q_i}\delta_{ij}$$

where $q_i = \frac{2}{3} - p_i$, and the p_i satisfy the Kasner conditions

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} {p_i}^2 = 1 \qquad (\Longrightarrow \quad \sum_{i=1}^{3} q_i = \sum_{i=1}^{3} {q_i}^2 = 1).$$

The density has the behaviour

$$\rho \approx \frac{1}{(t - t_0(\mathbf{x}))(t - t_1(\mathbf{x}))}.$$

These solutions behave just like the Bianchi type I cosmological solutions known as the Heckmann-Schücking solutions [26]. When the "bang-time" t_0 is constant the solutions either become vacuum-dominated and Kasner like (essentially the case discussed above) or they become matter-dominated and FRW-like in their behaviour.

Diagonal dust

The above discussions provide some fairly detailed descriptions of cosmological behaviour near the singularity, but one is left wondering exactly how general they are. In particular, is there a generic class of solutions which are either matter-dominated or curvature-dominated or both?

A detailed analysis of dust *without* assuming the metric to be velocity-dominated or vacuum-dominated [27] shows that in the diagonal case

$$ds^2 = -dt^2 + e^{2\alpha}dx^2 + e^{\beta}dy^2 + e^{\gamma}dz^2$$

where

$$\begin{aligned} e^{2\alpha} &\approx (t - t_0(\mathbf{x}))^{2p_1(\mathbf{x})} \\ e^{2\beta} &\approx (t - t_0(\mathbf{x}))^{2p_2(\mathbf{x})} \\ e^{2\gamma} &\approx (t - t_0(\mathbf{x}))^{2p_3(\mathbf{x})} \end{aligned}$$

the generic behaviour (generic meaning here that $\partial t_0/\partial x^i \neq 0$ in more than one direction) is $(p_1, p_2, p_3) = (0, 0, 1)$.

This example indicates that the generic behaviour of an inhomogeneous singularity is that it has quasi-regular behaviour such as that associated with a shell cross or emergence through a pancake. Such singularities are typically what we expect to happen *after* the big bang or "true" cosmological singularity. It seems to indicate that a strong enough matter singularity cannot be too strongly inhomogeneous.

3.3 Redshifted and blueshifted singularities

Another class of problems relates to the redshift properties of inhomogeneous models. As is well known, the light from particles of fluid emerging from a FRW model is infinitely redshifted, making this singularity relatively "benign". In the cases discussed above all models except those having FRW-like behaviour or having $(p_1, p_2, p_3) = (0, 0, 1)$ will display an *infinite blueshift* in one direction (the x^1-direction if we take $p_1 \leq p_2 \leq p_3$, since this implies $p_1 < 0$).

How are we to describe the redshift or blueshift properties of a cosmological model? The redshift is something which is not an intrinsic property of a space-time manifold, but is rather the property of specific vector fields on that manifold. A geometrical procedure for discussing redshifts may be carried out as follows[1].

Consider a space-time M enveloped by a larger manifold $\widehat{M}$, and let U be any open subset of M. Let v^μ be a unit timelike vector field ($v_\mu v^\mu = -1$) defined on U. At any point $p \in U$ let k^μ be a null vector, then we define the *Hubble index* of the vector field v^μ at p in the direction k^μ to be

$$H \stackrel{\text{def}}{=} \frac{v_{\mu;\nu} k^\mu k^\nu}{(v_\nu k^\nu)^2}.$$

The reason for the adoption of this name is as follows. Set $k^\mu = \alpha(v^\mu + e^\mu)$ where e^μ is a unit spacelike vector orthogonal to v^μ, i.e. $e_\nu v^\nu = 0$, $e_\nu e^\nu = 1$. It is a straightforward matter to show that the redshift at p in the direction e^μ over a short distance $\delta\ell$ is to first order

$$z = H\delta\ell,$$

Hubble's law with "Hubble constant" H. Of course there is nothing "constant" about H as it depends both on the position p and the direction e^μ, but the analogy is clear.

If we define the standard kinematic quantities shear, expansion, rotation and acceleration of the vector field in the usual way

$$v_{\mu;\nu} = \sigma_{\mu;\nu} + \frac{1}{3}\theta h_{\mu;\nu} + \omega_{\mu;\nu} - \dot{v}_\mu v_\nu$$

then

$$H = \theta_{\mu\nu} e^\mu e^\nu + \dot{v}_\mu e^\mu \tag{3}$$

where $\theta_{\mu\nu} = \sigma_{\mu;\nu} + \frac{1}{3}\theta h_{\mu;\nu}$.

The first term on the right hand side of equation (3) can be thought of as a purely Doppler shift, giving three principal Hubble constants θ_1, θ_2 and θ_3 defined by the three principal directions of the tensor $\theta_{\mu\nu}$. This is the entire redshift picture (blueshifts occurring if any of the θ_i are negative) for a geodesic vector field, $\dot{v}_\mu = 0$. It is interesting to note that the second term on the right hand side of equation (3) accounts for "gravitational redshifts" such as occur in the Schwarzschild solution when we take the vector field v^μ to be along the static Killing direction. It is not unreasonable then to think of the two terms on the right of equation (3) as being a Doppler shift and a gravitational shift respectively.

Consider now the case where U has an essentially singular boundary point $q \in \widehat{M}$, a representative of an abstract boundary point as discussed above. Choose the vector field to be the principal timelike direction of the energy-stress tensor (assuming it exists),

$$T^\mu_{\ \nu} v^\nu = -\rho v^\mu.$$

We will say then that the cosmological singularity $[q]$ has *bounded Hubble index* if along any integral curve $\gamma(t)$ of v^μ which approaches q as $t \to t_0$, the Hubble index H is bounded below on the sphere of directions around $\gamma(t)$ as $t \to t_0$. Otherwise we will say that the

[1]This is work done with K. Newman.

singularity has an *infinite blueshift*. The latter is the case that either $\theta_i \to -\infty$ for some i or the acceleration vector $\dot{v}_\mu$ becomes infinite in magnitude for some integral curve $\gamma(t) \to q$.

For a Kasner type metric

$$ds^2 \approx -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$$

we have for the geodesic vector field $v^\mu = (1, 0, 0, 0)$,

$$\theta_i \approx \frac{p_i}{t},$$

which has an infinite blueshift in general as $t \to 0$. The only exceptional case is $(p_1, p_2, p_3) = (0, 0, 1)$ which has bounded Hubble index. As discussed above, it appears that this is also the generic behaviour of diagonal dust. While diagonal dust is very special what the analysis seems to be telling us is that blueshifted singularities have some kind of inherent instability, and that the only models with infinite redshifts in all directions ("benign singularities") must be asymptotically FRW in their behaviour.

Another line of argument provided by Penrose [28], but based on thermodynamic principles also favours the idea that the universe should have an FRW-like origin. In Penrose's view it is the Weyl tensor which should have vanishingly small components (possibly related to a state of minimum entropy) near the cosmological singularity. Such a singularity is sometimes known as an *isotropic* or *conformal singularity*. Penrose's conjecture has come to be known as the *Weyl Curvature Hypothesis*. A considerable body of work has gone into showing that perfect fluid models which satisfy this hypothesis are constrained to be FRW [29, 30, 31, 32, 33]. A further natural question to ask is the following:

> Does an infinite redshift singularity imply that it is a conformal singularity, and consequently if it is a perfect fluid does it imply that the universe is at least in an asymptotic sense necessarily FRW?

4. Conclusions

Singularity avoidance and singularity acceptance are in a sense not totally exclusive strategies, at least not if one regards singularities as merely describing the asymptotic mathematical behaviour of space-time in the classical regime. Every theory has limits to its regime of validity, and general relativity has long been recognised as unlikely to be correct as one approaches the Planck era.

Singularity theory is still worth pursuing as it provides us with valuable information about our universe down to the quantum boundary. This boundary is a fuzzy area, perhaps characterised firstly by the breakdown of thermodynamics, as suggested by Bekenstein. As we proceed further into the pre-classical era, the continuum structure of space-time breaks down into some kind of discrete structure, but the physics is at present completely unknown. This preclassical era is possibly one of total chaos, but the emergence of such classical concepts such as *time* as a statistical quantity defined on the discrete structure [14], may leave the emergent classical world in a very ordered state. In accord with Penrose's ideas, this classical world is one of low entropy and must therefore be approximately FRW.

Time is probably the most interesting of the emergent quantities as we pass from quantum to classical physics. In the discrete era where neighbourhood-like elements connect with each other in a fairly random and unordered fashion (see figure 1) it probably has no existence at all. Subsequently it "exists most" when entropy is at its lowest at the beginning of the classical era, and again it loses all meaning in the final heat death into black holes and space-time re-enters a discrete epoch near the collapse singularity. In this scenario the history of our universe is indeed a history of time.

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