Quantum gravity: A brief review

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Abstract

A brief history of quantum gravity will be reviewed in this talk. Various aspects of quantum gravity ranging from the failure of perturbative treatments to the alternative formulations of quantum gravity such as supergravity and superstrings will be discussed at an informal level. The main emphasis of this review will be on the loop representation of non-perturbative quantum gravity.

1. Introduction

It is known since early this century that general relativity is incompatible with quantum theory. The incompatibility is indeed more profound than the fact that gravity is perturbatively non-renormalisable in the covariant quantisation scheme. It ultimately lies in the rôle in which space and time play in general relativity and quantum theory. This is a rather subtle issue and is undoubtedly the main culprit that defies various quantisation approaches to gravity. A less conceptually subtle issue is, of course, the non-renormalisability of gravity. This is more of a technical issue than a conceptual one. It arises from the attempt to depict gravity as another field defined on Minkowski space-time. Here, the problem encountered is primarily due to the presence of a dimensionful coupling constant—the Gravitational constant—(resulting from the Principle of Equivalence) that prevents the construction of a predictive quantum theory of gravity. Indeed, the advent of quantum field theory led invariably to valiant attempts in quantising Einstein's theory of gravitation. All of which proved futile. Perhaps Isham [25, p. 8] was on the right track all along when he remarked that rather than quantising gravity, one should seek a quantum theory which yields general relativity as its classical limit. But then, the main obstruction here is the lack of a starting point to construct such a quantum theory.

By assuming that quantum theory is the underlying principle governing the behaviour of nature at the fundamental level, it is then almost inevitable that a quantum theory of gravitation should exist.[1] Perhaps a more pertinent question to be raised at this juncture is the following: why quantise gravity in the first place? First, there are

[1] It will be assumed tacitly that quantum laws are the fundamental laws that govern at the microscopic level.

issues in quantum cosmology—such as the quantum effects of black holes due to their intense gravitational fields—which cannot be fully addressed without a consistent theory of quantum gravity. Second, it is hoped that a theory of quantum gravity will clear up various enigmatic questions such as the structure of space-time at a microscopic level, causality (and hence the arrow of time), and possibly even account for the presence of singularities in classical space-times [36, Chapter 8, p. 256] established by Hawking and Penrose. These questions provide rather strong incentives for constructing a theory of quantum gravity.

An early effort at quantising gravity was made by Rosenfeld in 1930 [48, 49]; needless to say, he headed rapidly into insurmountable technical difficulties! This is hardly surprising since it is now well known that pure gravity is perturbatively non-renormalisable at the 2-loop level and non-renormalisable at the 1-loop level when coupled with matter fields. Indeed, a simple power counting argument will quickly predict the non-renormalisability of gravity. In the early 1960's, Weinberg studied the quantum aspects of general relativity within the framework of S-matrix theory [61, 62], but his work was hindered by hideous non-linearities encountered in Einstein's field equations. His task was continued by Boulware and Deser [22] who showed in detail that, provided that the long range interactions of gravity are mediated by massless spin-2 particles, in the S-matrix formulation, general relativity is indeed the classical limit of the quantum theory. However, their calculations were done in the low-frequency domain.

In a paper by 't Hooft [57], it was demonstrated that pure gravity is 1-loop renormalisable but when coupled with matter, the theory ceases to make sense perturbatively. Specifically, Deser and Nieuwenhuizen showed that the Einstein-Maxwell fields diverge at the 1-loop level [27] and the quantised Einstein-Dirac system also diverges at the 1-loop level [26]. In a recent paper by van de Ven [58], the 2-loop non-renormalisability of covariant quantum gravity was proved explicitly. And to make matters even worse, aside from the technical issues of non-renormalisability, more conceptually profound questions posed—just to mention a few—by Wheeler regarding measurement [24, p. 224], and the issue of causality—*cf.* for example, references [15, 40]—must also be explained in a satisfactory manner by any candidate theory of quantum gravity.

An initial motivation for quantising gravity lay in the hope that it might eliminate the divergences that exist in quantum field theory—unfortunately, not only is such a hope dashed, but using perturbative methods gravity cannot be renormalised. This clearly suggests that the conventional means of quantising gravity, that is, the use of (perturbative) covariant quantisation, is not the right approach; or perhaps quantum theory is ultimately not a complete theory but merely an approximate theory describing the behaviour of nature at the fundamental level. Having said this much, this speculative note will not be pursued any further in this dissertation. However, the failure of gravity to be quantised perturbatively does not necessarily mean that a theory of quantum gravity fails to exist.

Quantum field theory demands that the background metric of space-time be fixed and that Poincaré-invariance be preserved.[2] Moreover, it assumes the smoothness of

[2] This is required in order for energy and momentum to be conserved locally.

the underlying space-time manifold. In quantum gravity, the metric itself becomes a dynamical variable and the gauge group is no longer the Poincaré group but the group of smooth diffeomorphisms. Also, it is worthwhile pointing out that quantum gravity, should it exist, ought to determine (or at least, predict) the structure of space-time at the Planck scale and below—assuming the smoothness of space-time certainly defeats this very purpose. Furthermore, the presence of quantum fluctuations of space-time geometry might well destroy its smooth structure. Indeed, a number of researchers in this field, Penrose [30, p. 4] or [47, p. 31] in particular, are quite convinced that the smoothness of space-time geometry at very small distances must be sacrificed. Some researchers go a step further and toy with the idea that perhaps even topology itself ought to be quantised, whatever such a statement might imply. At least, the motivation for such an observation is that *perhaps*, at the Planck scale, fluctuations in the spatial topology (of space-time) might occur, resulting in a space-time foam structure. For an account of space-time foams, refer to Hawking's paper [35]. Initial moves towards topological quantisation was initiated in a rigorous way by Isham *et al.* [39]. A rather eloquent (*and* convincing) argument outlining the need for a non-perturbative approach to gravity can be found in a monograph by Ashtekar [1, p. 3]; consult also references [55, §1], [31, p. 327] and [4].

2. Supergravity Theories

It should be pointed out that perturbative covariant quantisation of gravity (which failed to succeed anyway!) and the Ashtekar's quantisation programme are not the only means of tackling the problem of quantising gravity. There are others besides those two such as the Kaluza-Klein theory which currently seems to have gone out of favour amongst researchers working in the mainstream of quantum gravity. Probably the two most well known ones are supergravity and superstring theory. Incidentally, they were also candidates for a Unified Field theory. Curiously enough, string theory was originally conceived to provide an explanation for the behaviour of hadrons and not to quantise gravity!

Supersymmetry is the underlying principal ingredient in supergravity and superstrings. Roughly, it describes a transformation between bosonic fields and fermionic fields. Indeed, supersymmetry can only be implemented if space-time is curved! An heuristic argument outlining the equivalence between the presence of gravity and the implementation of local supersymmetry can be found in [59, p. 201]. This fact alone is suggestive that perhaps quantising gravity requires the unification of fundamental forces of nature. An excellent review article on supergravity can be found in reference [59].

In supergravity theories, each bosonic field has its fermionic counterpart (and vice versa). The fermionic partner of gravitational field is a spin $\frac{3}{2}$ field called the *gravitino*. If there are $n \leqq 8$ gravitinos, the theory is called an $N = n$ *supergravity* theory. $N = 0$ corresponds to general relativity theory. If $N > 8$, fields of spin $\frac{5}{2}$ (and higher) enter into the picture and this includes several spin 2 fields as well. However, the coupling of spin $\frac{5}{2}$ to gravity and to fields of different spins are known to be inconsistent, and no satisfactory coupling of fields with spins greater than 2 exists. Hence, N cannot be greater than 8.

In $N = 1$ supergravity theory, bosons and fermions (which occur in pairs) form irreducible representations of a supersymmetric algebra[3]—these are the spin $(2, \frac{1}{2})$ doublets (*i.e.*, the graviton-gravitino system), the spin $(1, \frac{1}{2})$ doublets (the photon-neutrino system) and the spin $(0, \frac{1}{2})$ doublets. It is a feature of the theory that as many matter doublets may be added to the spin $(2, \frac{3}{2})$ doublet as desired: in doing so, say, by adding one or more spin $(1, \frac{3}{2})$ doublets to spin $(2, \frac{3}{2})$ doublets, one obtains the extended $(N = 2, \ldots, 8)$ supergravity theories. These theories possess N Fermi-Bose symmetries (plus the usual space-time symmetries of course), $\frac{1}{2}N(N-1)$ spin 1 real vector fields and fields of lower spins. Moreover, they also have a global $U(N)$ group whereby the fermions rotate into themselves, and an $O(N)$ subgroup which rotates bosonic fields into themselves. In this way, the graviton—in N-extended supergravity theories—is replaced by a new superparticle whose "polarizations" yield gravitons, quarks, photons, gravitinos, leptons. This unification of particles into one superparticle leads to the unification of forces.

The ultra-violet divergences appearing in supergravity theories seem to be much better behaved. For instance, the infinities in the S-matrix in the first and second order quantum corrections cancel due to the symmetry between bosonic and fermionic fields. Nonetheless, even the presence of supersymmetry is not sufficient to guarantee finiteness at all loops—at least, there are no conclusive proofs that supergravity is perturbatively renormalisable [32]. Indeed, there are strong reasons to suspect that in 4-dimensional space-time, supergravity theories will diverge at the 3-loop level [43]. Hence, it too is not a particularly successful theory of quantum gravity. Moreover, only N-extended matter may be coupled to N-extended supergravity.

3. Superstring Theory

Superstrings paint a more optimistic picture than supergravity theories. However, one now requires a 10 dimensional space-time with supersymmetry built in. In spite of that, gravity is a necessary ingredient in order for a consistent quantum theory of superstrings to exist. From this viewpoint, strings as fundamental quanta are strongly supported by the presence of gravity. An introduction to Superstrings can be found in reference [25, p. 301] by Schwarz or Kaku [42]. Hitherto, it is the only candidate for a Unified Field Theory. Supergravity is now understood to be the low-energy limit of superstring theory. More on this matter will be broached in the next paragraph.

In the theory of Superstrings, the fundamental objects are extended 1-dimensional objects called *strings*. The strings can either be open (*i.e.*, a curve) or closed (*i.e.*, a loop). In short, this extension enables ultra-violet divergences appearing in the Feynman diagrams to be removed. There are two basic types of string theory: the *type I* superstring theory, wherein the strings are unoriented, and *type II* in which the strings

[3] Very briefly, this is an algebra with both commutation and anticommutation brackets.

are oriented. The latter is also known as *heterotic superstrings*. Type II closed superstring theories have $N = 2$ supersymmetry and hence contain $N = 8$ supergravity modelled on a 4-dimensional space-time as a limiting case. Informally, supergravity lies in the zero-mass sector of closed superstring theory. There, supergravity is quadratically divergent at the 1-loop level whereas its corresponding superstring theory is finite. Strings can interact by joining two ends (for open strings), or by breaking at an "interior" point (in the case of a loop) to form an open string. The latter is demanded by causality simply because two ends of a string must "decide" to interact at once without determining first whether they belong to the same string or not.

The inclusion of supersymmetry to string theory means that, aside from general relativity and Yang-Mills theory being included in it, supergravity and GUT are also included in this theory! However, in spite of such grandiose achievements, perturbative approach to superstring theory is plagued with problems [42, p. 285]. Only three major problems will be listed here:

(i) the low energy mass spectrum is still wrong;
(ii) the theory cannot select the true vacuum amongst the host of possible conformal field theories;
(iii) although supersymmetry is preserved to all orders in perturbative theory, it must be broken down in the low energy régime.

To address these problems, researchers turn towards a non-perturbative approach to superstring theory. Also, note that for bosonic string theory, the entire sum of the perturbative expansion diverges [33, 30]. The Ashtekar loop programme takes a more modest turn: it only seeks to formulate a consistent theory of quantum gravity without any thought of unifying the fundamental forces. And more importantly, the approach is non-perturbative from the outset! Indeed, the problems encountered by superstring theory, which is hitherto the sole candidate for a "proper" Unified Field theory, points towards a non-perturbative approach. A second important point to observe here is that the Ashtekar programme asserts that the gravitational field can be quantised on its own without any other fields, whereas in superstring theory, the very presence of supersymmetry necessitates the unification of forces in order to produce a consistent theory of quantum gravity. Quite a strong contrast indeed!

4. Non-perturbative Canonical Quantum Gravity

In this section, a cursory account of the canonical quantisation of gravity, together with the strengths and shortcomings of the Ashtekar quantisation programme, will be sketched. To condense the historical development of quantum gravity, it is enough to point out that from the late 1940's up to the mid-1950's, Bergmann embarked on a quest to canonically quantise field theories which are covariant under general coordinate transformations [18, 19, 20, 21]; here general relativity is of course a particular case those theories. He began by doing away with a space-time metric and considered instead a more fundamental field from which the Lagrangian of the theory was constructed. He quickly discovered that the system possessed constraints. Although his quantisation programme was not successfully completed, he nonetheless laid some important ground work for later researchers. In 1966, a comprehensive analysis of canonical quantum gravity was eventually carried out by DeWitt [28, 29].

In the canonical formalism of general relativity, covariance is violated and space-time is split into space and time. The resulting classical configuration space is the space of Riemannian 3-geometries with the cotangent bundle over the configuration space being the phase space of the system. It will suffice to note here that the resulting phase space of the gravitational system is constrained. That is, the physical trajectories in the phase space lie on a constraint surface defined by the Hamiltonian constraint and the diffeomorphism constraints. Upon canonically quantising this classical system, the physical states lie precisely in the kernel of both the quantum Hamiltonian and diffeomorphism constraint operators. In fact, this is only true for the case when the spatial 3-dimensional slice is chosen to be compact; in the non-compact case, the wavefunctionals must also satisfy an additional Schrödinger equation [25, Eqn (6.1.4), p. 79]. However, only the spatially compact case will be considered here. Unfortunately, due to the intractability of the quantum Hamiltonian constraint equation arising from involuted non-linearities, not a single explicit solution is known. This equation is known as the *Wheeler-DeWitt equation*,[4] and the wavefunctional that satisfies it is known broadly as the *wavefunction of the universe*.

Approximate solutions were of course found, but this involved truncating the Wheeler-DeWitt equation so that only a finite number of degrees of freedom are retained (instead of an infinite number of degrees of freedom in the full equation); this gave rise to the theory of baby universes—the *mini-superspace approximation*. At best, such solutions offer researchers a myopic insight into the convoluted nature of gravity. However, it should be remarked that even if the Wheeler-DeWitt equation can be solved, there remains the question of interpreting the solutions.

Loosely put, the wavefunctionals describe the physical states of space-time as probability amplitudes of possible histories. But this implies at once that the concept of time seems to have vanished from the picture; that is, there is the unpalatable absence of dynamics, of evolution, of time. This disturbing dissonance is seemingly overcome by identifying part of the geometry as an "intrinsic" time; then, the Wheeler-DeWitt equation is interpreted as encoding information that relates to how a wavefunctional changes with respect to this newly introduced notion of "time". But alas, by introducing a physical inner product on the Hilbert space of physical states, the integration integrates over "time" as well! Hence, the problem of time is really not resolved at all. Time, however it might be interpreted here, is treated very differently from quantum theory. See Isham [25, §6, p. 78] for a lucid but laconic account relating to the problem of time in this canonical formulation and other related problems arising from quantising in the canonical formalism.

It should be pointed out that the riddle of timelessness only occurs for the spatially compact case. When the spatial slice of space-time is non-compact, time is defined by a

[4] More accurately, the sum of the diffeomorphism and Hamiltonian constraint equation is known as the *Wheeler-DeWitt equation*.

Schrödinger evolution equation. For a lively assessment of the canonical approach, refer to [31, §2, p. 330] by Ashtekar. Before concluding this sorry tale, a brief word must be mentioned on the Hartle-Hawking functional integral approach to Wheeler-DeWitt equation. Aside from commenting that it yields, heuristically at least,[5] explicit solutions to the Wheeler-DeWitt equation, it fails to provide any information whatsoever at the instant of creation. Also, there is the confounded issue of time cropping up time and time again! It is thus a fervent hope that the problem of time will be resolved with the formulation of a consistent theory of quantum gravity.

If (2+1)-quantum gravity was not mentioned earlier, then it is simply because it is essentially an open book! Much work has been done on it. In particular, (2+1)-quantum gravity is often used as a toy-model for the seemingly intractable (3+1)-quantum gravity. For more details, see for example reference [63] by Witten—as well as a complementary paper by Moncrief [45] who made some constructive criticisms regarding the conclusions drawn by Witten in his paper—and more recent ones such as [44, 11], or a somewhat refreshing article by Waelbroeck [60] to name just a few out of the plethora of literatures on (2+1)-quantum gravity.

5. The Ashtekar-Rovelli-Smolin Quantisation Programme

In this section, a very brief summary of Ashtekar's Hamiltonian formulation of general relativity [2, 3] and the loop representation theory of quantum gravity [52] will be sketched. The motivation for the Ashtekar formulation of general relativity is to attempt to simplify and hence solve the Hamiltonian constraint equation arising from the traditional ADM formalism. Without delving into great detail, fix a smooth, compact 3-manifold Σ and consider a Lorentzian 4-manifold $M \cong \Sigma \times \mathbb{R}$, where Σ defines a co-dimension 1 spacelike foliation on M. The Ashtekar connection 1-form A is a complexified $\mathfrak{su}^{\mathbb{C}}(2)$-connection 1-form given by

$$A_a(x)_A{}^B = \Gamma_a(x)_A{}^B - \frac{\mathrm{i}}{\sqrt{2}}\Pi_a(x)_A{}^B,$$

where Γ_a is the SU(2) spin-connection coefficients compatible with a triad E on Σ and K_a relates to the extrinsic curvature of Σ on the constraint surface by $K_{ab} = -\mathrm{tr}\,\Pi_{(a}E_{b)}$. The canonical conjugate to A is a densitised SU(2) triad $\tilde{E} = \sqrt{\det q}E$ of weight 1 on Σ. They satisfy the following Poisson bracket relation:

$$\{A_a(x)^{AB}, \tilde{E}^b(y)_{CD}\} \stackrel{\text{def}}{=} -\frac{\mathrm{i}}{\sqrt{2}}\delta_a^b\delta_C{}^{(A}\delta_D^{B)}\delta^3(x,y).$$

Recall that a triad relates to a Riemannian 3-metric q on Σ by $\det q \cdot q_{ab} = -\mathrm{tr}\,(\tilde{E}_a \cdot \tilde{E}_b)$.

In terms of this new pair of variables $(A, \tilde{E})$, the constraint equations of general relativity take the following form:

$$C_1(A,\tilde{E}) = \mathcal{D}_a\tilde{E}^a = 0, \tag{5.1}$$

$$C_2(A,\tilde{E}) = \mathrm{tr}\,(\tilde{E}^aF_{ab}) = 0, \tag{5.2}$$

$$C_3(A,\tilde{E}) = \mathrm{tr}\,(\tilde{E}^a\tilde{E}^bF_{ab}) = 0 \tag{5.3}$$

[5] The infinite-dimensional measure involved in the integral is not rigorously defined.

where $\mathcal{D}$ is the covariant derivative induced by the Ashtekar connection A: $\mathcal{D}_a\psi_M = \partial_a\psi_M + GA_aM^N\psi_N$, G is the Gravitational constant and $F_{abM}{}^N$ is the curvature 2-form of $A_{aM}{}^N$. In all that follows, units will be chosen so that $G \equiv 1$.

Quantising these new variables gives rise to the *self-dual representation*, where the state vectors are holomorphic functionals $\psi = \psi[A]$ on the space of Ashtekar connection 1-forms. The quantised pair $(A, \tilde{E})$ is defined by

$$A_a(x) \to \hat{A}_a(x), \quad \tilde{E}^a(x) \to -\mathrm{i}\frac{\delta}{\delta A_a(x)},$$

where $\hat{A}_a(x)$ acts on functionals $\psi = \psi[A]$ on the space of Ashtekar 1-forms by multiplication. In brief, Ashtekar's formulation of complex general relativity [3] leads immediately to the connection representation of quantum gravity—the general relativity formulated in [3] is really "real" general relativity in terms of a *complex* and a real variable, the Ashtekar connection and its conjugate momentum respectively.

An advantage of formulating general relativity in terms of connections (the Ashtekar connection 1-forms) and their conjugates—these are the soldering forms; *i.e.*, "square roots" of metrics—is that that the conjugate variable need not be invertible! See equations (5.1)–(5.3). This differs greatly from general relativity which demands that the metric be non-degenerate. An obvious conclusion to be drawn from Ashtekar's formulation is that it yields solutions that are more general than those obtained via Einstein's field equations. It is perhaps a somewhat tantalising speculation that Ashtekar's formulation will yield a profound insight into the relation between signature changes in the space-time metric and the changes in spatial topology of space-time, and perhaps even more interestingly, how these affect quantum gravity. An instructive preliminary analysis regarding spatial topological changes and the degeneracies of Lorentzian metrics can be found in an article by Horowitz [38]. A related comment, if somewhat premature at this stage as it pertains to the loop representation to be mentioned shortly below, relates to an intriguing paper by Smolin [56]: he demonstrated that, using the loop representation of quantum gravity, the spatial topological changes effected by creating or annihilating a special class of wormholes—what he calls *minimalist wormholes*, which are created by identifying pairs of distinct points on the spatial 3-manifold—is equivalent to general relativity coupled to a single Weyl fermion field!

Another positive spin-off from Ashtekar's formulation of general relativity is that in the connection representation, the Hamiltonian constraint is greatly simplified—indeed, to the extent that some nontrivial solutions can now be found: they are just the Wilson loops. Unfortunately, Wilson loops are not invariant under diffeomorphisms. For more details, see [46, p. 12–13] or [41, §7, p. 333]. This startling hitch led to the development of the loop representation of quantum gravity by Rovelli and Smolin [52]. In the loop representation, solutions to *all* the quantum constraints were found—refer to [41, 52] again.

The loop representation of quantum gravity involves the introduction of a set of classical T^n-observables $T^{a_1\cdots a_n}[\gamma, A, \tilde{E}](s^1, \ldots, s^n)$ defined as follows:

$$\operatorname{tr}(U_{\gamma,A}(s^2, s^1)\tilde{E}^{a_1}(\gamma(s^1))U_{\gamma,A}(s^3, s^2)\tilde{E}^{a_2}(\gamma(s^2))\ldots U_{\gamma,A}(s^1, s^n)\tilde{E}^{a_n}(\gamma(s^n))),$$

where $U_{\gamma,A}(t,s) \stackrel{\text{def}}{=} \mathcal{P}e^{\oint_{\gamma(s)}^{\gamma(t)} A}$ is the spinor propagator of A along γ from $\gamma(s)$ to $\gamma(t)$, $\mathcal{P}$ denotes the path-ordering operator and $T^0[\gamma, A] \stackrel{\text{def}}{=} \operatorname{tr} U_{\gamma,A}$. The set of these T-observables has the following Poisson bracket structure: $\{T^n, T^m\} \sim T^{n+m-1}$, and the Poisson brackets reflect how the loops combine and break at each point where $\tilde{E}$ is attached. The Poisson brackets are actually singular in the sense that each term involves a δ-distribution of the form

$$\Delta^a[\gamma,\eta](s) = \int_0^1 \mathrm{d}t\, \delta^3(\gamma(s), \eta(t))\dot{\eta}^a(t).$$

These however turn out to be harmless and they can be smeared away.

In the loop representation, these classical T-observables become linear operators on the space of multi-loop functionals. The T^0-observable acts on a multi-loop functional Ψ in the following way: $(\hat{T}^0[\gamma, A]\Psi)(\eta) = \Psi[\gamma \cup \eta]$, where γ, η are loops in Σ. The quantum T^n-operators act on loops by breaking and joining them all of the points where $\tilde{E}$ is attached simultaneously. The details can be found in [52].

Some remarks are due below. First, because the T-observables are SU(2) gauge invariant—due to the presence of the trace operator—the Gauss constraint, equation (5.1), present in the connection representation is eliminated automatically on passing into the loop representation. Second, the diffeomorphism constraint, equation (5.2), and the Hamiltonian constraint, equation (5.3), can be derived from the T-observables by defining a suitable limiting procedure whereby the loop contracts down to a point. Third, non-trivial solutions to both the Hamiltonian and diffeomorphism constraints in the loop representation theory can be solved. And what is more, the theory predicts a discrete structure at the Planck scale. More of this will be said in the next section.

The loop representation theory was applied to free Maxwell theory with resounding success [12]. It was later applied to linearised quantum gravity [13] and was shown to correctly reproduce gravitons. Applications were also made to (2+1)-dimensional quantum gravity primarily on tori [44] using the connection as well as the loop representation—the Dirac transformation reveals that they are all equivalent. In the case of (2+1)-quantum gravity, the loop representation yields a combinatorial picture whereas the connection representation depicts a "timeless" one. Of course, going over to (3+1)-quantum gravity is a different matter altogether. There are no local degrees of freedom in (2+1)-dimensions (due to the vanishing of Weyl tensor), whereas this is no longer the case with (3+1)-gravity. For other work on (2+1)-quantum gravity in the loop representation, refer to papers by Ashtekar *et al.* [6, 11].

6. Discussion

Unfortunately, like most theories in the real world, the loop representation of quantum gravity is not free of problems. There are a number of unresolved issues. One of the problems of the loop representation was discovered by Brügmann and Pullin [23, §4, p. 239]. They noticed with some consternation that solutions of the quantum Hamiltonian operator represented by products of Wilson loops were also annihilated by a metric determinant operator in terms of the Ashtekar variables. It follows as a corollary that the solutions will also satisfy the Hamiltonian constraints for arbitrary cosmological constant! A concise account can be found in [46, p. 13–14].

Another disturbing problem of the loop representation lies in the physical interpretation of the theory. Attempts have been made at interpreting the theory in terms of knots and weaves by Rovelli and Smolin [53, 10]. See also references [64, 65 66] by Zegwaard. Also, a physical inner product on the multi-loop states is not known: this is a problem that is intimately tied with the physical interpretation of the theory. Moreover, there is the pressing issue of defining physical observables [40, 51, 5]. Once again, all of these issues are intertwined; plus, the fact that very little is known about classical observables in general relativity does very little by way of lighting a path for ardent researchers. In fact, there is also a minor technical problem with the definition of the space of classical T-observables. Although the space of T-observables supports a Poisson structure, the space lacks a linear structure altogether—this can be easily seen by taking two distinct loops γ, η and two points s, t such that $\gamma(s) \neq \eta(t)$. Then, $T^1[\gamma, A, E](s) + T^1[\eta, A, E](t)$ is not even defined in the space of classical T-observables. In short, it is not an algebra. Indeed, with a bit of tedious algebra, it can be shown that the Poisson brackets do not satisfy the Jacobi identity. Fortunately, it can be shown that the space of classical T-observables can be imbedded in a suitable linear extension. The details can be found in reference [68]. It is also shown in the same reference that the algebra of quantum T-operators is a covering space of the suitable linear extension of the classical T-observables.

In spite of the various setbacks encountered, Smolin [53, 54] has constructed a number of interesting observables in quantum gravity: a surface area operator, a volume operator and an operator that measures the "length" of a 1-form on the spatial slice of space-time. The spectacular results arising from the first two operators are that area and volume in quantum gravity are quantised in some multiple of the Planck area and Planck volume respectively! This seems to vindicate the conjecture that the structure of space-time is discrete at the Planck scale—a conjecture that was established heuristically by Rovelli [67, §4, p. 1648]. Along this note, Rovelli and Smolin [54] also constructed a physical Hamiltonian operator (with a cosmological term included) which acts in essence by breaking and rejoining the points of intersections of loops in different ways. Moreover, it is also finite as well as diffeomorphism-invariant. Hope is expressed that the Hamiltonian operator might encode the full contents of Einstein's field equations in a diffeomorphism-invariant manner.

Returning to other obstacles present in the theory, there are technical matters such as the construction of a measure on the space of Ashtekar connection 1-forms—preliminary studies have been made by Ashtekar *et al.* [7, 8, 9, 16][6] and by Baez [16, 17]. The construction of an explicit diffeomorphism-invariant measure on the multi-loop space as well as expressing the reality conditions in the loop variables are issues that need addressing: the absence of a physical inner product to date are related directly to these problems. On a whole, the future to the Ashtekar quantisation programme is

[6] They constructed a diffeomorphism-invariant promeasure on the space of connections.

not as bleak as it seems, and aside from its mathematical beauty, it is at present, a novel approach towards a non-perturbative quantum gravity that has yet to reach an *impasse*. Indeed, recently, further progress in the connection representation is made. Ashtekar *et al.* [10] performed a detailed study of diffeomorphism-invariant theories in the connection representation and they found complete solutions to the Gauss and diffeomorphism constraints for the following class of theories in the connection representation: the Husain-Kuchař model, Riemannian general relativity and Chern-Simons theories. Furthermore, they were able to endow the space of such states with a Hilbert space structure, where the inner product of the Hilbert space is compatible with the reality conditions imposed on the theories.

References

1. Ashtekar, A., *Lectures On Non-Perturbative Canonical Gravity*, World Scientific, Singapore, 1991.
2. ———, *New variables for classical and quantum gravity*, Phys. Rev. Letts. **57** (1986), 2244–2247.
3. ———, *New Hamiltonian formulation for general relativity*, Phys. Rev. D **36** (1987), 1587–1602.
4. ———, *Recent mathematical developments in quantum general relativity*, LANL archives preprint gr-qc/9411055 (1994).
5. ———, *On the relation between classical and quantum observables*, Commun. Math. Phys. **71** (1980), 59–64.
6. Ashtekar, A., Husain, V., Rovelli, C. and Smolin, L., *2+1 gravity as a toy model for the 3+1 theory*, Class. Quantum Grav. **6** (1989), L185–L193.
7. Ashtekar, A. and Isham, C., *Representations of the holonomy algebras of gravity and non-Abelian gauge theories*, Class. Quantum Grav. (1922), 1433–1467.
8. Ashtekar, A. and Lewandowski, J., *Projective techniques and functional integration for gauge theories*, Pennsylvania preprint, CGPG-94/10-6 (1994).
9. Ashtekar, A., Lewandowski, J., Marolf, D., Mourão and Thiemann, T., *Coherent state transforms for spaces of connections*, Pennsylvania preprint, CGPG-94/12-2 (1994).
10. ———, *Quantization of diffeomorphism invariant theories of connections with local degrees of freedom*, LANL archives preprint gr-qc/9504018 (1995).
11. Ashtekar, A. and Loll, R., *New loop representations for 2+1 gravity*, Pennsylvania preprint CGPR-94/5-1 (1994).
12. Ashtekar, A. and Rovelli, C., *A loop representation for the quantum Maxwell field*, Class. Quantum Grav. **9** (1992), 1121–1150.
13. Ashtekar, A., Rovelli, C. and Smolin, L., *Gravitons and loops*, Phys. Rev. D **44** (1991), 1740–1755.
14. ———, *Weaving a classical metric with quantum threads*, Phys. Rev. Lett. **69** (1992), 237–240.
15. Ashtekar, A. and Stachel, J. (editors), *Conceptual Problems Of Quantum Gravity, 1988 Osgood Hill Conference*, Birkhäuser, Boston, 1991.

16. Baez, J., *Generalized measures in gauge theory*, LANL archives preprint hep-th/9310201 (1993).
17. ______, *Link invariant, holonomy algebra and functional integrations*, LANL archives preprint hep-th/9301063 (1993).
18. Bergmann, P., *Non-linear field theories*, Phys. Rev. **75** (1949), 680–685.
19. ______, *The Hamiltonian of general theory of relativity with electromagnetic field*, Phys. Rev. **80** (1950), 81–88.
20. ______, *Classical and quantum field theories in Lagrangian formalism*, Phys. Rev. **89** (1953), 4–16.
21. ______, *Dirac bracket transformations in phase space*, Phys. Rev. **98** (1955), 531–538 and p. 544.
22. Boulware, D. and Deser, S., *Classical general relativity derived from quantum gravity*, Ann. Phys. **89** (1975), 193–240.
23. Brügmann, B. and Pullin, J., *Intersecting N loop solutions of Hamiltonian constraints of quantum gravity*, Nucl. Phys. B **363** (1991), 221–244.
24. Christensen, S. (editor), *Quantum Theory Of Gravity*, Adam Hilger, Bristol, 1984.
25. Davies, A. and Sutherland, D (editors), *Superstrings And Supergravity (proceedings of 28th Scottish Universities Summer School in Physics)*, SUSSP, Oxford, 1986.
26. Deser, S. and van Nieuwenhuizen, P., *One-loop divergences of quantised Einstein-Maxwell fields*, Phys. Rev. D **10** (1974), 401–410.
27. ______, *Non-renormalisability of quantised Dirac-Einstein system*, Phys. Rev. D **10** (1974), 411–420.
28. DeWitt, B., *Quantum theory of gravity I: the canonical theory*, Phys. Rev. **160** (1967), 1113–1148.
29. ______, *Quantum theory of gravity II: the manifest covariant theory, & III: applications of the covariant theory*, Phys. Rev. **162** (1967), 1195–1256.
30. Duff, M. and Isham, C. (editors), *Quantum Structure Of Space And Time*, Cambridge University Press, Cambridge, 1982.
31. Ehlers, J. and Friedrich, H. (editors), *Canonical Gravity: From Classical to Quantum (proceedings, Bad Honnef, Germany, 1993)*, LNP 434, Springer-Verlag, Berlin, 1994.
32. Grisaru, M, and Siegel, W., *Supergraphity II: manifest covariant rules and higher-loop finiteness*, Nucl. Phys. B **201** (1982), 292–314.
33. Gross, D, and Periwal, V., *String perturbation theory diverges*, Phys. Rev. Lett. **60** (1988), 2105–2108.
34. ______, *Gross and Periwal reply*, Phys. Rev. Lett. **61** (1988), 1517.
35. Hawking, S., *Space-time foam*, Nucl. Phys. B **144** (1978), 349–362.
36. Hawking, S. and Ellis, G., *The Large Scale Structure Of Space-Time*, Cambridge University Press, Cambridge, 1973.
37. Hawking, S. and Israel, W. (editors), *General Relativity: An Einstein Centenary Survey*, Cambridge University Press, Cambridge, 1979.
38. Horowitz, G., *Topology change in classical and quantum gravity*, Class. Quantum

Grav. **8** (1991), 587–601.

39. Isham, C., Kubyshin, Y. and Renteln, P., *Quantum norm theory and the quantisation of metric topology*, Class. Quantum Grav. **7** (1990), 1053–1074.
40. Isham, C., Penrose, R. and Sciama, D. (editors), *Quantum Gravity, vols I & II*, Oxford University Press, Oxford, 1982.
41. Jacobson, T. and Smolin, L., *Nonperturbative quantum geometries*, Nucl. Phys. B **299** (1988), 295–345.
42. Kaku, M, *Strings, Conformal Fields And Topology: An Introduction*, CP, Springer-Verlag, Berlin, 1991.
43. Marcus, N., and Sagnotti, A., *The ultraviolet behaviour of $N = 4$ Yang-Mills and the power counting of extended superspace*, Nucl. Phys. B **256** (1985), 77-108.
44. Marolf, D., *Loop representation for 2+1 gravity on a torus*, Class. Quantum Grav. **10** (1933), 2625–2647.
45. Moncrief, V., *Reduction of the Einstein equations in 2+1 dimensions to a Hamiltonian system over Teichmüller space*, J. Math. Phys. **30** (1989), 2907–2914.
46. Nicolai, H. and Matschull, H.-J., *Aspects of canonical gravity and supergravity*, Hamburg preprint DESY 92-099 (invited contribution to the Karpacs Winter School) (1992).
47. Penrose, R., *Nonlinear gravitons and curved Twistor theory*, Gen. Rel. & Grav. **7** (1976), 31–52.
48. Rosenfeld, L., *Zur Quantelung der Wallenfelder*, Ann. Physik **5** (1930), 113.
49. ———, *Über die gravitationswirkungen des lichtes*, Z. Physik **65** (1930), 589.
50. Rovelli, C., *What is observable in classical and quantum gravity?*, Class. Quantum Grav. **8** (1991), 297–316.
51. ———, *Quantum reference systems*, Class. Quantum Grav. **8** (1991), 317–331.
52. Rovelli, C. and Smolin, L., *Loop space representation of quantum general relativity*, Nucl. Phys. B **331** (1990), 80–152.
53. ———, *Knot theory and quantum gravity*, Phys. Rev. Lett. **61** (1988), 1155–1158.
54. ———, *The physical Hamiltonian in non-perturbative quantum gravity*, LANL archives preprint gr-qc/9308002 (1993).
55. Smolin, L., *Recent developments in non-perturbative quantum gravity*, LANL archives preprint hep-th/9202022 (1992).
56. ———, *Fermions and topology*, LANL archives preprint gr-qc/9404010 (1994).
57. 't Hooft, G., *An algorithm for the poles at dimension four in dimensional regularization procedure*, Nucl. Phys. B **62** (1973), 444–460.
58. van de Ven, A., *Two-loop quantum gravity*, Nucl. Phys. B **378** (1992), 309–366.
59. van Nieuwenhuiszen, P., *Supergravity*, Phys. Rep. **68** (1981), 189–398.
60. Waelbroeck, H., *Solving the time-evolution problem in 2+1 gravity*, Nucl. Phys. B **364** (1991), 475–494.
61. Weinberg, S., *Photons and gravitons in perturbation theory: derivation of Maxwell's and Einstein's equations*, Phys. Rev. B **138**, 988–1002.
62. ———, *Photons and gravitons in S-matrix theory: derivation of charge conserva-*

tion and equality of gravitational and inertial mass, Phys. Rev. B **135** (1964), 1049–1056.

63. Witten, E., *2+1 dimensional gravity as an exactly soluble system*, Nucl. Phys. B **311** (1988), 46–78.
64. Zegwaard, J., *Physical interpretation of loop representation for non-perturbative quantum gravity*, Class. Quantum Grav. **10** (1993), S273–S276.
65. ______, *Gravitons in loop quantum gravity*, Nucl. Phys. B **378** (1992), 288–308.
66. ______, *The weaving of curved geometries*, LANL archives preprint hep-th/9210033 (1992).
67. Rovelli, C., *Ashtekar formulation of general relativity and loop space non-perturbative quantum gravity: a report*, Class. Quantum Grav. **8** (1991), 1613–1675.
68. Toh, T.-C., *The loop algebra of quantum gravity*, (submitted).