Holonomies in quantum gravity

E.E. Wood
Department of Physics, Monash University,
Clayton, Vic. 3168, Australia

Abstract

The holonomy loop representation has been utilised in $3+1$ canonical quantum gravity so as to provide solutions to the canonical constraint equations. These solutions have a particularly elegant form in that they are solutions within knot classes. Motivated by this we propose a generalisation of the holonomy loop approach. To introduce this new approach we employ a more general notion of a Lie gauge group, namely a Lie gauge groupoid. We discuss the mathematical properties possessed by a field theory with groupoid as opposed to group structure and briefly discuss it's advantages and hence our hopes for its applications in solving the problem of quantum gravity.

1. Overview of quantum gravity

The limitations of modern quantum field theory are most apparent in its inability to model a quantum theory of gravity. This failure manifests itself in the theory's non-renormalisability and is directly related to the perturbative techniques associated with quantum field theory. As such nonperturbative approaches seem to be the main candidates for enabling the construction of a quantum theory of gravity. The most notable non-perturbative technique is the holonomy loop representation [1] associated with Ashtekar's general relativity [2]. Unfortunately this nonperturbative and holonomy dependent approach has yet to produce a nonperturbative model of even the most simplest interacting quantum field theory, such as quantum electrodynamics. Moreover, it has technical problems, e.g., ill-defined measure, lack of inner product etc, that have not been solved since the theory's inception.

As such we are motivated to propose an alternative holonomy dependent formulation of interacting field theories and general relativity, which exploits a little know symmetry called groupoid symmetry. Groupoids provide a convenient way to embody the multiple symmetries present in interacting field theories and lead to a path integral formalism of the quantum theory.

2. Ashtekar's general relativity

Ashtekar's general relativity is an alternative formulation of Einstein's general relativity. The motivation behind such an alternative formalism can be summarised as follows:

– It provides an unexpected link between general relativity and gauge field theory, in that the physical field is not the metric but rather a gauge field. As such it sets the groundwork for unifying general relativity with the other forces.

– A direct consequence of its gauge nature is that it has lead to the introduction of gauge techniques into general relativity. The most profound example of this is the loop representation, utilisation of which leads to a set of solutions to the constraint equation of the theory.

– It provides an important mathematical generalisation of general relativity which has lead to an important extension of the standard Dirac quantisation procedure in such a manner that reality conditions should lead to a well defined inner product on the space of physical states. Moreover, its generalised formalism results in degenerate and non-degenerate metric solutions of the equations of motion. A, as yet, unexplored consequence of this result is with regards to its application to spacetime singularities.

In distinction to the Einstein-Hilbert action associated with general relativity the action of Ashtekar's general relativity [2] takes on a Yang-Mills flavour as follows

$$S[e,A] = \int d^4x\, e_{\mu I} e_{\nu J} F^{IJ}{}_{\tau\sigma}\epsilon^{\mu\nu\tau\sigma} \tag{1}$$

in which $e^I{}_\mu$ is a tetrad, (i.e., four linearly independent vector fields), $\epsilon^{\mu\nu\tau\sigma}$ is a totally antisymmetric tensor, and $F^{IJ}{}_{\mu\nu} = \partial_\mu A^{IJ}{}_\nu - \partial_\nu A^{IJ}{}_\mu + A^{IM}{}_\mu A^J{}_{M\nu} - A^{IM}{}_\nu A^J{}_{M\mu}$ is the field strength associated with the self dual gauge field connection $A^{MN}{}_\mu = -\frac{i}{2}\epsilon^{MN}{}_{IJ}A^{IJ}{}_\mu$, which is an element of $SO(3,1) \simeq SL(2;\mathbb{C})$. The associated equations of motion are

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} e_{\nu J} F^{IJ}{}_{\rho\sigma} &= 0, & (2)\\ (\delta^{KI}\delta^{LJ} + \frac{i}{2}\epsilon^{KLIJ})\epsilon^{\mu\nu\rho\sigma} D_\rho(e_{\mu I} e_{\nu J}) &= 0, & (3) \end{aligned}$$

where D is the covariant derivative. A consequence of the above formalism is that the metric is no longer a fundamental physical quantity but rather it is a derived quantity. One can show that if $(e_\mu{}^I, A^{IJ}{}_\mu)$ satisfy the above equations of motion, then the spacetime metric which is related to the tetrads by

$$g_{\mu\nu} = e_\mu{}^I e_\nu{}^J \eta_{IJ}\,, \tag{4}$$

is in fact a solution to the Einstein vacuum equations. (Here η_{IJ} is the flat Minkowski metric.)

A canonical ADM–type formulation is obtained by introducing variables

$$\begin{aligned} N^a &= e^a{}_I e^{0I} & (5)\\ N &= \frac{e}{\sqrt{q}} & (6)\\ E^a{}_I &= \sqrt{q}\left(e^a{}_I - N^a e^0{}_I\right) & (7) \end{aligned}$$

where $a = 1,2,3$ is purely spatial, and q is the determinant of the 3-metric $q_{ab} \equiv g_{ab}[e]$. The action (1) may be rewritten [3]

$$S[E, N, N^a, e^0{}_I] = \int d^4x \, \{iA^i{}_a[e]\dot{E}^a_i + iA^i{}_0[e]C_i + iN^aC_a + NC\}\,. \tag{8}$$

in the (freely chosen) gauge $e^0{}_i = 0$, $i = 1,2,3$, where

$$\begin{aligned} C_i &= D_b E^b{}_i, & (9)\\ C_a &= E^b{}_i F^i{}_{ab}, & (10)\\ C &= E^a{}_i E^b{}_j F^{ij}{}_{ab}. & (11) \end{aligned}$$

The only dynamical variable is the physical triad $E^a{}_i$. The variables $\{N, N^a, e^0{}_i\}$ are Lagrange multipliers, leading to the constraints

$$C_i[A,E] = 0, \qquad C_a[A,E] = 0, \qquad C[A,E] = 0. \tag{12}$$

Let $\gamma(s)$ be a parameterised curve in the 3-dimensional physical space with coordinates $\gamma^a(s)$. The Wilson loop is defined by

$$T[\gamma] = \mathrm{Tr}\; U_\gamma \tag{13}$$

in which $U_\gamma(s) = \mathcal{P}\exp(-ig\int_\gamma A_a d\gamma^a)$ is the *holonomy* associated with the gauge connection $A_a = A^i{}_a\tau_i$ along $\gamma(s)$. Here τ_i are the Pauli matrices, i represents the internal gauge group indices and $\mathcal{P}$ means *path-ordered*, in a sense defined in [3]. The holonomy is a solution to the differential equation defining parallel transport

$$\frac{dU}{d\gamma^a} = -igA_aU\,. \tag{14}$$

The curve $\gamma(s)$ is most commonly taken to be closed loop, $s \in [0, 2\pi]$, due to conventional formulations of the reconstruction theorem [4]. The above differential equation is subject to the boundary condition $U_\gamma(0) = I$ where I is the identity element of the gauge group at $\gamma(0)$.

Inserting the physical triad field $E^a{}_i$ associated with the ADM version of Ashtekar's general relativity into the loop associated with the holonomy the following quantity is obtained

$$T^a[\gamma](s) = \mathrm{Tr}\;(U_\gamma(s)E^a{}_i(\gamma(s))\tau^i) \tag{15}$$

The two holonomy dependent quantities T and T^a are observables in the ADM canonical phase space $(A^i{}_a, E^a{}_i)$. The diffeomorphism and Hamiltonian constraint equations expressed in terms of such holonomy dependent observables reduce to

$$\begin{aligned} C_a &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\epsilon_{abc}T^b[\gamma_{x,c,\varepsilon}](0) & (16)\\ C &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\epsilon_{abc}T^{ab}[\gamma_{x,c,\varepsilon}](0,\varepsilon^2) & (17) \end{aligned}$$

where $\gamma_{x,c,\varepsilon}$ is a loop centred in x, lying in the ab plane (normal to the c direction) with area ε, ϵ_{abc} is the 3-dimensional volume form and[1]

$$T^{ab}[\gamma](s,t) = \mathrm{Tr}\ (U_\gamma(t,s)E^a U_\gamma(t,s+2\pi)E^b) \tag{18}$$

is a generalisation of T^a. (These results are derived by making use of the expansion $U \approx I + \varepsilon F_{ab}\epsilon^{abc} + \mathrm{O}(\varepsilon)$ for the holonomy U.)

The classical phase space $(A^i{}_a, E^a{}_i)$ is consequently replaced with loop space in the holonomy approach. Functionals on such a loop phase space can be obtained from functionals on $(A^i{}_a, E^a{}_i)$ via the holonomy loop representation [1]

$$\Psi[\gamma] = \int d\mu[A]\ U_\gamma\ \Psi[A] \tag{19}$$

where $d\mu[A]$ denotes a measure on the gauge connection space. A vector space V can be formed from such functionals $|\Psi[\gamma]\rangle \in V$ to construct a physical sector of V of states which are annhilated when the classical constraints (12) are elevated to operator constraints

$$\hat{C}_a|\Psi[\gamma]\rangle = 0, \tag{20}$$
$$\hat{C}|\Psi[\gamma]\rangle = 0. \tag{21}$$

The general form of the solution is

$$|\Psi[\gamma]\rangle = |\Psi[K[\gamma]]\rangle \tag{22}$$

where $K[\gamma]$ the generalised knot class of γ in the spatial three manifold [5].

The loop representation utilised in Ashtekar's general relativity has had substantial successes [3], but also has some a number of limitations. One serious drawback of this formalism is that as yet an inner product, $\langle\Psi_1|\Psi_2\rangle$, on the space of physical states has not been given, due to the ill-defined nature of the measure in (19) – a problem common to other approaches to quantum gravity. In a limited sense this problem has been overcome in $2+1$ dimensional general relativity in which the measure is definable in term of link invariants [6]. Except for this and other simple cases, such as electromagnetism [7], the loop formulation is not well understood. Furthermore, the approach is intrinsically canonical and as such artificially treats space and time differently, which is not in the spirit of relativity. Due to these drawbacks we are motivated to propose an alternative formalism in the following section utilising an alternative symmetry to gauge group symmetry called gauge groupoid symmetry.

3. Groupoid formulation

A Yang-Mills theory is a gauge field theory, that is, a theory in which the symmetries of the theory are encoded into a group, from which a principle fibre bundle $P(M,G)$ can

[1] $U_\gamma(t,s+2\pi)$ is the parallel transport from t to the origin and then from the origin to s.

be constructed given a base manifold M. For an interacting theory the gauge group is a product of groups, for example electroweak theory with gauge group

$$SU(2) \times U(1)$$

or for gauge and matter interactions in quantum electrodynamics

$$U(1) \times SL(2;\mathbb{C})$$

The associated principle fibre bundle

$$P(M, G_1 \times G_2)$$

of such interacting field theories can alternatively be expressed as a groupoid [8]. The are several advantages of a groupoid formalism over that of a principle fibre bundle largely due to the more general nature of the groupoid over that of the bundle description. For reviews on groupoids and there potential applications to gauge theory see [9] and [10] and references therein.

Briefly, a *groupoid* Ω over a base X is a set (Φ, X) where Φ are the set of isomorphic maps. Φ are the "elements" of the groupoid Ω. In the literature Φ is sometimes called the groupoid – however, in the present paper the groupoid will be denoted by Ω. X is a set of objects. The groupoid has two projection maps α,$\beta : \Phi \to X$, called respectively the source and target maps, and one object inclusion map $\epsilon : X \to \Phi$. The main difference between groups and groupoids is that only a partial composition is defined for groupoids, that is,

$$z_1.z_2 : \Phi.\Phi \to \Phi \text{ for } z_1 \text{ and } z_2 \text{ elements of } \Phi.$$

The composition satisfies the following conditions
(1) $\Phi.\Phi = \{z_1.z_2 \in \Phi.\Phi \text{ such that } \beta(z_1) = \alpha(z_2)\}$.
(2) If $\beta(z_1) = \alpha(z_2)$ and $\beta(z_2) = \alpha(z_3)$ then $z_1.(z_2.z_3) = (z_1.z_2).z_3$.
(3) Each $z \in \Omega$ has an inverse z^{-1} such that $\alpha(z^{-1}) = \beta(z)$ and $\beta(z^{-1}) = \alpha(z)$. Moreover, the inverse has the feature that $z.z^{-1} = \epsilon(\alpha(z))$ is a left identity element, and $z^{-1}.z = \epsilon(\beta(z))$ is a right identity element.
(4) The left identity element satisfies $\epsilon(\alpha(z)).z = z$ and the right identity element satisfies $z.\epsilon(\beta(z)) = z$.

Consider a groupoid Ω, with fixed point $x \in X$, that is

$$\Omega_x = \{z \in \Phi : \text{ such that } \alpha(z) = x\}.$$

Now $\{\Omega_x\}_{x \in X}$,which denotes the collection of all Ω_x as x varies over the base X, is a principal fibre bundle. The Ω_x are the fibres which are isomorphic to the structure group. Thus a principal fibre bundle is actually a groupoid with a fibred point over which the group structure is defined. The bundle projection map, which is usually denoted by π is the α map.

The starting point for the work which we are investigating is to construct a groupoid Wilson line in 4-dimensional spacetime,

$$\Phi[\gamma] = \mathcal{P}\exp(-ig\int A_\mu d\gamma^\mu) \tag{23}$$

in which γ is now an *open* as opposed to a *closed* path. The groupoid Wilson line is gauge groupoid invariant without taking its trace, and thus avoids the subsequent information loss associated with the conventional Wilson loop [11].

In distinction to the canonical nature in which the holonomies enter into Ashtekar's general relativity we exploit the intrinsic path dependent nature associated with the groupoid to construct a Feynman path integral in terms of groupoid Wilson lines. The advantages of this approach are as follows:

– other fundamental interaction such as electroweak forces and the strong force are easily introduced in combination with general relativity due to the natural capacity inherent in groupoids to describe multiple group symmetries.

– within the groupoid formalism quantum groups naturally occur, due to the open structure of the nonlocal constituent variables. This reinforces the intimate relation between quantum groups and knot structure discovered in the holonomy loop representation of general relativity.

– quantisation is via path integrals and as such avoids the artificial spacetime split inherent in the canonical approach.

4. Conclusion

Some of the areas we plan to further investigate with the proposed groupoid formalism are the construction of nonperturbative versions of quantum field theories such as quantum electrodynamics, electroweak theory and especially quantum gravity. Whether or not the generalised gauge theory embodied via groupoids has experimentally verifiable results is currently being explored in the context of the Casimir effect and the Lamb shift.

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