The gravitational effects of cusps on cosmic strings

Malcolm R. Anderson
Department of Mathematics, Edith Cowan University,
WA 6050, Australia

Abstract

The trajectories of idealised (zero-thickness) cosmic strings in flat space-time typically contain isolated points, known as cusps, where the local radius of curvature of the string goes to zero. It has long been known that the weak-field approximation breaks down in the vicinity of a cusp, leading to a beam of gravitational radiation directed parallel to the motion of the cusp. In this paper I show that the weak-field approximation also breaks down in a region with radius of order $(G\mu)^2 L$ and enclosed mass of order $(G\mu)M$, where L is the length of the string, M is its total mass, and μ is its mass per unit length. I further indicate how a self-consistent analysis of the effects of a cusp within the full framework of general relativity might be performed.

1. Introduction

Cosmic strings are long filaments of false vacuum energy that may have formed during the strong-electroweak phase transition, which is thought to have occurred some 10^{-35} seconds after the Big Bang. The likely cosmogonic effects of cosmic strings were first examined in the early 1980s, and the success of the earliest simulations encouraged the belief that string loops could help explain the formation of large-scale structure in the Universe. Since then, more detailed numerical work has largely discredited the assumptions which underpinned the simplest and most convincing of the string-seeded cosmologies, and nowadays there is very little active work being done in the area. Nonetheless, cosmic strings are not entirely devoid of interest. Their gravitational properties, in particular, are rich and counter-intuitive, and they have helped shed some light on some of the more obscure aspects of general relativity.

Strictly speaking, cosmic strings are conventionally modelled as vortex solutions of the Einstein-Higgs-Yang-Mills field equations, with action

$$S = \int d^4x \sqrt{-g}\, [\, |(\partial_\mu - ieA_\mu)\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\lambda(|\phi|^2 - \eta^2)^2\,], \tag{1}$$

where ϕ is a complex scalar and A_μ a real vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, e, λ and η are adjustable constants, and g is the determinant of the background 4-metric $g_{\mu\nu}$. Vortices of this type are often referred to as "local strings". If the constants are chosen to correspond to a Grand Unified phase transition, the resulting cosmic string has a thickness of about 10^{-30} cm and a mass per unit length μ of about 10^{22} g/cm.

Since the thickness of a local string is much smaller than any length scales of astrophysical interest, it is common to approximate cosmic strings as line singularities in space, giving rise to what are known as "idealised" cosmic strings. The history of an idealised cosmic string is a timelike 2-surface (or "world sheet") with parametric representation

$$x^\mu = X^\mu(\zeta^A),$$

where the index A runs from 0 to 1. The 2-metric induced on the surface is:

$$\gamma_{AB} = g_{\mu\nu}\, X^\mu{}_{,A}\, X^\nu{}_{,B}$$

and is assumed to have signature (+,-). Förster [1] has shown that, if the thickness of the vortex is small in comparison with the local radius of curvature, then the Nielsen-Olesen action [1] reduces to the Nambu-Goto action:

$$S = -\mu \int d^2\zeta\, \sqrt{-\gamma}, \tag{2}$$

where $\gamma = \det(\gamma_{AB})$ and the constant μ can be interpreted as the mass per unit length of the string.

The equations of motion corresponding to the idealised action [2] are:

$$\gamma^{AB} D_B\, X^\mu{}_{,A} = 0, \tag{3}$$

where the world-sheet derivative operator D_B is defined by:

$$D_B\, X^\mu{}_{,A} = X^\mu{}_{,AB} + \Gamma^\mu_{\kappa\lambda}\, X^\kappa{}_{,A}\, X^\lambda{}_{,B} - \Gamma^C_{AB}\, X^\mu{}_{,C}\,,$$

with $\Gamma^\mu_{\kappa\lambda}$ and Γ^C_{AB} the connections generated by $g_{\mu\nu}$ and γ_{AB} respectively. Also, the action [2] gives rise, in the usual way, to a distributional energy-momentum tensor of the form:

$$T^{\mu\nu}(x) = \frac{\mu}{\sqrt{-g}} \int d^2\zeta\, \sqrt{-\gamma}\, \gamma^{AB}\, X^\mu{}_{,A}\, X^\nu{}_{,B}\, \delta^4(x-X). \tag{4}$$

The idealised string is therefore characterised by a constant energy per unit length and a tension of equal magnitude μ.

In flat space-time, the equations of motion [3] can in principle be solved exactly. A generic feature of almost all loop solutions in flat space-time is that at certain points on the world sheet the tangent 2-space is degenerate. In physical terms, the string is instantaneously travelling at the speed of light at such a point: the local radius of curvature is then zero, and the string forms what is known as a "cusp". Cusps can have dramatic effects on the gravitational properties of a cosmic string. In the weak-field approximation, the gravitational field diverges in the neighbourhood of a cusp, and there is an additional divergent effect (sometimes referred to as "gravitational beaming") on the null line projected forward from the cusp in the direction of its motion.

In view of the breakdown of the weak-field approximation, and also the fact that the approximations leading to the Nambu-Goto action [2] are inoperative near a cusp, it has been suggested that field-theoretic and gravitational effects – either separately or together – will act to suppress the formation of cusps. In this paper I review what little is known about the gravitational aspect of this problem. In particular, I examine in some detail the gravitational properties of cusps at the level of the weak-field approximation, and also suggest how the problem of gravitational back-reaction might be addressed in a fully self-consistent manner.

2. The dynamics of strings in flat space-time

In Minkowski space-time, it is always possible to choose the world-sheet coordinates $\zeta^0 = \tau$ and $\zeta^1 = \sigma$ so that they satisfy the "standard gauge" conditions:

$$X,_\tau^2 + X,_\sigma^2 = 0 \tag{5}$$

and

$$X,_\tau \cdot X,_\sigma = 0. \tag{6}$$

The equations of motion [3] then reduce to the 2-dimensional wave equation:

$$X^\mu,_{\tau\tau} = X^\mu,_{\sigma\sigma} \tag{7}$$

with general solution

$$X^\mu = [\tau, \frac{1}{2}\{\mathbf{a}(\tau+\sigma) + \mathbf{b}(\tau-\sigma)\}], \tag{8}$$

where $\mathbf{a}$ and $\mathbf{b}$ are 3-vector functions which are arbitrary except for the gauge constraints

$$|\mathbf{a}'| = |\mathbf{b}'| = 1,$$

which follow directly from [5] and [6]. [Note that in deriving the solution [8] residual gauge freedom was used to set τ equal to the Minkowski time coordinate x^0.]

The induced 2-metric corresponding to the general solution [8] is:

$$\gamma_{AB} = \frac{1}{2}(1 - \mathbf{a}' \cdot \mathbf{b}')\,\mathrm{diag}(1, -1). \tag{9}$$

Cusps occur whenever $\mathbf{a}' \cdot \mathbf{b}' = 1$; that is, whenever $\mathbf{a}'$ and $\mathbf{b}'$ coincide. The 2-metric [9] is then degenerate, and the world-sheet tangent vectors are

$$X^\mu,_\tau = [1, \mathbf{a}']$$

and

$$X^\mu,_\sigma = [0, \mathbf{0}],$$

so that the velocity of the string is instantaneously null. Furthermore, the intrinsic curvature of the world sheet,

$$R^{(2)} = -8(1 - \mathbf{a}' \cdot \mathbf{b}')^{-3}[\mathbf{a}'' \cdot \mathbf{b}''(1 - \mathbf{a}' \cdot \mathbf{b}') + \mathbf{a}'' \cdot \mathbf{b}'\,\mathbf{a}' \cdot \mathbf{b}'']$$

is divergent at a cusp.

A simple method for analysing the dynamics of a string in flat space-time has been developed by Kibble and Turok [2, 3]. Since $\mathbf{a}'$ and $\mathbf{b}'$ are unit vectors, they separately trace out curves on the surface of the unit sphere as τ and σ vary. A cusp will occur whenever the two curves cross.

If the string forms a closed loop, the position vector X^μ and the tangent vectors $X^\mu{}_{,\tau}$ and $X^\mu{}_{,\sigma}$ are periodic functions of the spacelike coordinate σ with some period L. This in turn means that $\mathbf{a}'$ and $\mathbf{b}'$ are also periodic with period L, and so trace out closed curves on the unit sphere. Moreover, since $\mathbf{a}$ and $\mathbf{b}$ are periodic, the centroids of $\mathbf{a}'$ and $\mathbf{b}'$ must lie at the origin of the sphere, and so the curves cannot stay in a single hemisphere. Unless the curves traced out by $\mathbf{a}'$ and $\mathbf{b}'$ are relatively convoluted they will cross at least twice, giving rise to at least two recurrent cusps during each period of oscillation of the string.

Cusps are therefore in some sense generic to string loops. Indeed, it has even been suggested that the effect of gravitational radiation from a string loop would be to suppress the higher-order harmonics in the curves traced out by $\mathbf{a}'$ and $\mathbf{b}'$ and so enhance the possibility of cusp formation. However, this argument is somewhat tendentious, as gravitational radiation is typically generated by regions of high intrinsic curvature (as, for example, near cusps) rather than by harmonics on the Kibble-Turok sphere.

3. Cusps in the weak-field approximation

In the harmonic gauge, the deviation $h_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$ is, to linear order, the retarded potential solution of the weak-field Einstein equations:

$$h_{\mu\nu}(t,\mathbf{x}) = 4G\int d^3x'\,|\mathbf{x}-\mathbf{x}'|^{-1}S_{\mu\nu}(t',\mathbf{x}'), \tag{10}$$

where the source function $S_{\mu\nu}$ is a linear functional of the energy-momentum tensor $T_{\mu\nu}$:

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda$$

and $t' = t - |\mathbf{x}-\mathbf{x}'|$ is the retarded time.

In the case of an idealised cosmic string satisfying the flat-space equation of motion [7], the expression [4] for the energy-momentum tensor gives:

$$S_{\mu\nu}(t',\mathbf{x}') = \mu\int d\sigma\, F_{\mu\nu}(t',\sigma)\,\delta^3(\mathbf{x}'-\mathbf{q}(t',\sigma)), \tag{11}$$

where

$$F_{\mu\nu} = \dot{X}_\mu\,\dot{X}_\nu - X'_\mu\,X'_\nu - \eta_{\mu\nu}\,\dot{X}^2,$$

$X^\mu(\tau,\sigma) = [\tau,\mathbf{q}(\tau,\sigma)]$ is the position vector [8] on the world sheet, an overdot denotes $\partial/\partial\tau$, a prime denotes $\partial/\partial\sigma$, and

$$\mathbf{q}(\tau,\sigma) = \frac{1}{2}[\mathbf{a}(\tau+\sigma)+\mathbf{b}(\tau-\sigma)].$$

If the expression [11] for $S_{\mu\nu}$ is substituted into [10], the retarded potential becomes:

$$h_{\mu\nu}(t,\mathbf{x}) = 4G\mu \int d\sigma\,(1-\mathbf{n}\cdot\dot{\mathbf{q}})^{-1}|\mathbf{x}-\mathbf{q}|^{-1}F_{\mu\nu}(\tau,\sigma), \tag{12}$$

where

$$\mathbf{n} = |\mathbf{x}-\mathbf{q}|^{-1}(\mathbf{x}-\mathbf{q})$$

is the unit vector from the source point $\mathbf{q}(\tau,\sigma)$ on the string to the field point $\mathbf{x}$, and the parametric time τ is a function of t, $\mathbf{x}$ and σ given implicitly by the equation

$$\tau = t - |\mathbf{x}-\mathbf{q}(\tau,\sigma)|.$$

Note here that the only points which contribute to $h_{\mu\nu}(t,\mathbf{x})$ are those on the intersection of the world sheet with the backward light cone of $[t,\mathbf{x}]$. The additional factor $(1-\mathbf{n}\cdot\dot{\mathbf{q}})^{-1}$ appears in [12] because the retarded time t' in the distributional factor $\delta^3(\mathbf{x}'-\mathbf{q}(t',\sigma))$ in [11] is itself a function of $\mathbf{x}'$.

At a cusp, $\dot{\mathbf{q}}^2 = 1$ and $F_{\mu\nu} = \dot{X}_\mu\,\dot{X}_\nu$, where $\dot{X}_\mu$ is null. Hence, the integrand in the expression [12] for the retarded potential $h_{\mu\nu}$ has a pole when $\mathbf{n}\cdot\dot{\mathbf{q}} = 1$; that is, when the field point $\mathbf{x}$ lies on the forward null cone of the cusp in the direction of the cusp's instantaneous velocity. Field points at which $\mathbf{n}\cdot\dot{\mathbf{q}} = 1$ define what is sometimes called the "gravitational beam" of the cusp. Near the beam, integration of equation [12] gives:

$$h_{\mu\nu}(t,\mathbf{x}) \sim G\,\mu\,\dot{X}_\mu\,\dot{X}_\nu\,|\mathbf{x}-\mathbf{q}_0|^{-1}(t-|\mathbf{x}-\mathbf{q}_0|)^{-1/3}R^{4/3}, \tag{13}$$

where $\mathbf{q}_0$ is the position vector of the cusp, and

$$R = |\mathbf{a}_0'' + \mathbf{b}_0''|^{-1}$$

is a length scale associated with the cusp, and is typically of the order of the length L of the loop. The expression [13] for the potential $h_{\mu\nu}$ near the beam was first derived by Vachaspati [4], and for an asymmetric loop suggests that the beaming of gravitational radiation from cusps would quickly accelerate the loop to relativistic velocities. In fact, at the level of the weak-field approximation, gravitational beaming from cusps accounts for a large fraction of the total gravitational energy radiated by loops in numerical simulations of the evolution of string networks [5].

Less well-advertised than gravitational beaming is the fact that the weak-field approximation also breaks down at points on the forward light cone of a cusp away from its beam. If $|\mathbf{x}-\mathbf{q}_0|$ is small but $\mathbf{n}\cdot\dot{\mathbf{q}}_0 \ll 1$ then equation [12] gives:

$$\begin{aligned} h_{\mu\nu}(t,\mathbf{x}) &\sim G\,\mu\,\dot{X}_\mu\,\dot{X}_\nu \int \frac{d\sigma}{|\mathbf{x}-\mathbf{q}_0|-\tau} \\ &\sim G\,\mu\,\dot{X}_\mu\,\dot{X}_\nu\,|\mathbf{x}-\mathbf{q}_0|^{-1/2}\,R^{1/2}. \end{aligned} \tag{14}$$

In other words, if the cusp is approached from any direction other than along the beam, the potential diverges as $r^{-1/2}$, where r is the spatial distance to the cusp. This indicates that the weak-field approximation breaks down not only near the beam but also inside a radius $r \sim (G\mu)^2 R$ about the cusp (where $G\mu \sim 10^{-6}$ for a GUT string).

The result embodied in equation [14] can be derived in a more heuristic fashion, as follows. If the cusp lies at $\tau = \sigma = 0$ then the source points near the cusp have the parametric form:

$$\mathbf{q}(\tau, \sigma) \sim \mathbf{q}_0 + \dot{\mathbf{q}}_0 \tau + \frac{1}{4}[\mathbf{a}_0''(\tau + \sigma)^2 + \mathbf{b}_0''(\tau - \sigma)^2],$$

and so at time $\tau = 0$

$$|\mathbf{q} - \mathbf{q}_0| \sim \frac{1}{4} R^{-1} \sigma^2.$$

Since μ is the rest mass per unit length of the string, and σ in the standard gauge measures the proper length of the string, the mass M inside a radius $|\mathbf{q} - \mathbf{q}_0| = r$ is:

$$M \sim 2\,\mu\,|\sigma| \sim 4\,\mu\,(rR)^{1/2}$$

and hence

$$G\,M\,/\,r \sim G\,\mu\,(R\,/\,r)^{1/2}. \tag{15}$$

According to [15], the potential again diverges as $r^{-1/2}$ and the weak-field approximation breaks down when $r \sim (G\mu)^2 R$. Furthermore, the mass M_C inside the strong-field region is predicted to be

$$M_C \sim 4\,G\,\mu^2 R \sim 4\,G\,\mu\,M_S,$$

where $M_S = \mu L \sim \mu R$ is the total mass of the string. For a GUT string, therefore, about 10^{-6} of the total mass of the string would be contained in the near-cusp region. Since the total mass of a string loop with length of order of the current horizon radius would be comparable to the mass of a large galaxy, the cusp mass M_C is not necessarily negligible.

The breakdown of the weak-field approximation near a cusp probably indicates that something more complex than mere gravitational beaming occurs there. On the face of it, there would seem to be two alternative fates for a cusp on a cosmic string: either higher-order corrections to the Nambu-Goto action [2] suppress the formation of a full cusp, with the result that the gravitational field of the string departs only minimally from the weak-field approximation; or the cusp is unstable to strong-field effects, and fundamentally new features appear (including perhaps the collapse of the cusp to form a black hole).

Unfortunately, neither alternative can at present be ruled out. For example, if field-theoretic effects were to act to limit the local Lorentz factor of the string to a maximum value γ then an analysis similar to that which led to equation [15] indicates that the potential at a distance r from a source point with Lorentz factor γ would be:

$$G\,M\,/\,r \sim G\,\mu\,(R\,/\,r)^{1/2}\,[1 - \gamma^{-1}(R\,/\,r)^{1/2}\,]. \tag{16}$$

According to [16], the weak-field approximation will still break down when $r \sim (G\mu)^2 R$, provided that γ is larger than $(G\mu)^{-1} \sim 10^6$. However, it is precisely for Lorentz factors of the order of $(G\mu)^{-1}$ that the zero-thickness idealisation central to the derivation of the Nambu-Goto action breaks down.

In the absence of any definite information about the consequences of field-theoretic corrections for the formation of cusps, it is natural to proceed on the assumption that they do form, and to attempt to calculate their gravitational effects within the full framework of general relativity. This is a complex task in itself, and at present it is only possible to indicate how the calculation might be performed.

4. Cusps in general relativity

There are very few known exact gravitating solutions with the string energy-momentum tensor [4]. Those solutions that have been published invariably describe space-times with a high degree of symmetry. They include the basic straight-string metric first derived (independently) by Gott and Hiscock in 1984 [6, 7]:

$$ds^2 = dt^2 - dz^2 - dr^2 - (1 - 4G\mu)^2 \, r^2 \, d\phi^2 \tag{17}$$

(where t, r, ϕ and z are standard cylindrical polar coordinates on $\Re^4$), as well as elaborations which describe a straight string interacting with plane-fronted gravitational (or "travelling") waves [8], with cylindrical gravitational waves [9, 10, 11, 12, 13, 14], and with a non-rotating black hole [15]. In fact, the only known strong-field result dealing with a string loop (rather than an infinite straight string) is Hawking's 1990 proof that a circular loop of cosmic string will collapse to form a black hole with a loss of at most 29% of its energy [16].

Furthermore, the viability of the distributional description of a cosmic string embodied in the energy-momentum tensor [4] has been strongly called into question by Geroch and Traschen [17]. The objections they level against the distributional description are many and varied, and it is not my intention to discuss them in detail here. However, the salient points in their argument are as follows:

1. The Einstein equations are non-linear equations in the metric tensor g_{ij} and therefore do not admit a natural interpretation as equations on distributions. Of course, in practice it is not the metric tensor that is a pure distribution, but rather the Riemann curvature tensor R_{ijkl}. Geroch and Traschen define a class of "regular metrics" which they feel is the broadest possible class of metrics admitting distributional curvature. It turns out that the metric [17] due to an infinite straight string in the distributional approximation is not regular. Furthermore, Geroch and Traschen claim to have proved that no regular metric exists on a 4-dimensional spacetime with curvature concentrated on surfaces of dimension 2 or smaller. They therefore conclude that it is doubtful that cosmic strings can meaningfully be described in terms of distributions.

2. There is little point in replacing a thin source of stress-energy such as a cosmic string with a source concentrated on some suitably-chosen surface unless there is a well-defined relationship between the original source and its distributional approximation. Gott [6], Hiscock [7] and Linet [18] have constructed families of metrics describing gravitating cylinders that reduce to the standard distributional string metric in the limit of zero radius, and in each case there is a simple linear relationship between the mass per unit length of the cylinders and the angle deficit induced

by the distribution. Yet all of these solutions satisfy a specialised equation of state, and Geroch and Traschen rightly point out that small deviations in the equation of state typically lead to a much more complicated relationship between the mass per unit length and the angle deficit. Also, in the absence of isometries (as for example if the string is curved) it is not clear that there exists a suitable relationship between source and field at all, as the Killing fields used in defining the mass per unit length and the angle deficit are no longer available.

3. An even more serious objection relates to the locally-flat nature of the metric exterior to an infinite straight string in the distributional approximation. The class of cylindrically-symmetric vacuum metrics forms a two-parameter family (the Levi-Civita family), and the standard distributional string metric [17] fills only a one-parameter subclass. Again, Geroch and Traschen point out that a small deviation in the equation of state of the source fluid can generate an exterior metric which is not locally flat and has quite a different asymptotic structure.

I have attempted to rebut these objections in detail elsewhere [19]. The most important points to note are that:

- the straight string metric [17] does have a well-defined distributional Einstein tensor $R_{ij} - \frac{1}{2}R\,g_{ij}$, even though it does not belong to Geroch and Traschen's class of "regular metrics";
- the metric [17] forms a very special subclass of the Levi-Civita family in the sense that any continuous sequence of static cylinders of perfect fluid *with bounded energy per unit length* will converge to [17] in the limit as the outer radius of the cylinders goes to zero; and
- even in the absence of Killing fields it is possible to define the angle deficit of a zero-thickness cosmic string and so (at least locally) relate the gravitational field of a curved string to its rest mass per unit length μ.

It is the third point that is relevant to a self-consistent treatment of the gravitational field near a string cusp. It turns out that the angle deficit $\Delta\theta$ (as measured by the Gauss-Bonnet formula) along any closed curve encircling the world sheet of the straight string metric [17] is extremal if the curve lies entirely within a surface of constant t and z. Hence, the surfaces of constant t and z can be distinguished in a manner which does not rely on the existence of Killing fields. In the case of a curved string, each point $\mathbf{p}$ on the world sheet of the string has associated with it a "normal" surface $\mathbf{N_p}$, defined to be the unique geodesically-generated spacelike surface through $\mathbf{p}$ which extremises the angle deficit $\Delta\theta$. If the world sheet is parameterised by coordinates ζ^A $(A = 0, 1)$ then each point on the normal surface $\mathbf{N_p}$ out to the local radius of curvature of the string (where the normal surfaces begin to cross) can be assigned coordinates $[\zeta^A(\mathbf{p}), r, \phi]$, where r is the geodesic distance along $\mathbf{N_p}$ from $\mathbf{p}$, and ϕ is an angle coordinate on $\mathbf{N_p}$.

If the angle deficit $\Delta\theta$ is specified to have the same value (in fact $8\pi G\mu$) on each of the normal surfaces, the metric in the neighbourhood of the string world sheet can be expressed in the 3+1 form:

$$ds^2 = g_{AB}\, d\zeta^A\, d\zeta^B + 2\, g_{A\phi}\, \zeta^A\, d\phi + g_{\phi\phi}\, d\phi^2 - dr^2.$$

Furthermore, the world sheet parameters $(\zeta^0, \zeta^1) = (\tau, \sigma)$ can always be chosen so that

$$g_{AB} = \kappa\, \eta_{AB} = \kappa\, \mathrm{diag}(1, -1)$$

on the world sheet, where κ is some undetermined function of τ and σ.

For small values of the geodesic radius r, the vacuum Einstein equations admit solutions for which the metric functions g_{AB}, $g_{A\phi}$ and $g_{\phi\phi}$ are multinomial expansions in r^2 and $r^{1/\alpha}$, where $\alpha = 1 - 4G\mu < 1$ is a constant. The leading-order behaviour of these expansions is as follows:

$$\begin{aligned} g_{AB} \;&=\; \kappa\, [\, \eta_{AB} + (\, L_{AB}\, \cos\phi + M_{AB}\, \sin\phi\,)\, r^{1/\alpha} \\ &\quad + (\, \frac{1}{4}\, R^{(2)}\, \eta_{AB} - \kappa^{-1} \alpha^{-2}\, \omega_A\, \omega_B\,)\, r^2\,] + O(r^{2+1/\alpha}), \end{aligned}$$

$$\begin{aligned} g_{A\phi} \;&=\; r^2\, \omega_A + (2\alpha+1)^{-1} [\, \omega^B\, (\, L_{BA}\, \cos\phi + M_{BA}\, \sin\phi\,) \\ &\quad + \alpha^2\, \kappa^{-1} (\, \kappa\, L^B_A\,)_{,B}\, \sin\phi - \alpha^2\, \kappa^{-1} (\, \kappa\, M^B_A\,)_{,B}\, \cos\phi\,]\, r^{2+1/\alpha} + O(r^4) \end{aligned}$$

and

$$g_{\phi\phi} = -\, \alpha^2\, r^2\, (\, 1 - \frac{1}{6}\, R^{(2)}\, r^2\,) + O(r^{2+2/\alpha}), \tag{18}$$

where the twist vector ω_A, the symmetric travelling wave potentials L_{AB} and M_{AB}, and the intrinsic curvature function $R^{(2)} = \kappa^{-1} \eta_{AB} (\ln \kappa)_{,AB}$ are all functions of τ and σ alone. Also, the travelling wave potentials satisfy the trace conditions:

$$\eta^{AB}\, L_{AB} = \eta^{AB}\, M_{AB} = 0.$$

Details of the derivation of the expansions [18] can be found in [20]. For present purposes, it seems worthwhile to examine their implications for the behaviour of the gravitational field near a cusp. At a cusp, the intrinsic curvature $R^{(2)}$ of the world sheet diverges, and so the metric expansions are presumably divergent as well. Nonetheless, it might still be possible to use the expansions to investigate the gravitational field in the vicinity of a spacelike cross-section of the world sheet with high, but finite, intrinsic curvature (that is, immediately prior to the formation of a cusp). In particular, I will here address the question of whether a closed trapped surface forms in the neighbourhood of a cusp. If so, then the cusp will inevitably give rise to some sort of singularity in the metric [21].

Consider therefore a closed quasi-spherical surface $\mathbf{T}$ on which τ is constant, parameterised by the equation $r = r(\sigma)$. The future-directed null normals on $\mathbf{T}$ have the general form $m_\mu \pm n_\mu$, where

$$m_\mu = [1 - g^{\sigma\sigma} r'^2,\; g^{\tau\sigma} r'^2,\; 0,\; -g^{\tau\sigma} r']$$

and

$$n_\mu = \lambda\,[0,\, r',\, 0,\, -1]$$

with

$$\lambda = \sqrt{(g^{\tau\sigma} r')^2 + g^{\tau\tau}(1 - g^{\sigma\sigma} r'^2)}.$$

Here, r' denotes $dr/d\sigma$.

The surface $\mathbf{T}$ will be trapped if the expansion of both the null normals is negative everywhere on $\mathbf{T}$. This in turn is equivalent to the condition

$$(h^{\mu\nu} m_{\mu;\nu})^2 - (h^{\mu\nu} n_{\mu;\nu})^2 > 0, \tag{19}$$

where $h^{\mu\nu}$ is the projection operator on $\mathbf{T}$. Suppose that a cusp forms on the world sheet at $\zeta^A = (\tau, \sigma) = (0, 0)$. Then for small values of $|\zeta|$ it turns out that $\kappa \sim |\zeta|^2$ and $R^{(2)} \sim |\zeta|^{-4}$ if the world sheet is adequately described by a flat-space solution of the form [8]. If the surface $\mathbf{T}$ is chosen so that $r(\sigma) \leq |\zeta|^2$ then the product $R^{(2)} r^2$ occurring in the metric expansions [18] remains bounded. After a somewhat lengthy and tedious calculation, it can be shown that the terms $(h^{\mu\nu} m_{\mu;\nu})^2$ and $(h^{\mu\nu} n_{\mu;\nu})^2$ appearing in the trapping condition [19] are both of order r^{-3}. At this level of approximation, therefore, it is not possible to prove or disprove that a closed trapped surface will form. Nonetheless, the calculation does confirm that there is a significant deviation from flat space near a cusp: for a quasi-spherical surface of constant t in Minkowski space-time, $(h^{\mu\nu} m_{\mu;\nu})^2$ is identically zero and $(h^{\mu\nu} n_{\mu;\nu})^2$ is typically of order r^{-2}.

5. Conclusions

In this paper it has been possible to give only a brief taste of the problems to be faced in developing a fully self-consistent treatment of the gravitational field of a cosmic string, particularly in the neighbourhood of a cusp. I have shown that the weak-field approximation breaks down everywhere in the vicinity of a cusp rather than (as was previously thought) just along the beam. This breakdown, it is true, is a weak one, in the sense that the gravitational potential diverges as $r^{-1/2}$, but for a GUT string the region in which strong-field effects appear will typically have a radius of the order of 10^{-12} times the total length of the string and an enclosed mass of the order of 10^{-6} times the mass of the string, which is large enough to be cosmogonically interesting.

What happens to a cosmic string when a cusp threatens to form remains an open question. The two most likely possibilities – namely that field-theoretic effects suppress the formation of the cusp and the weak-field approximation remains viable, or that strong-field effects dominate and lead, perhaps, to the formation of a singularity – cannot at present be ruled out. However, I am hopeful that the metric expansions [18], or some elaboration of them, will prove to be useful tools for analysing the gravitational field of a string.

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