Debye potentials for the massless Dirac equation in algebraically special spacetimes*

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Abstract

In a class of algebraically special spacetimes, we show that a solution of a complex scalar wave equation leads to a solution of the massless Dirac equation. The converse is also true. This technique allows us to construct a symmetry operator for the massless Dirac equation using a conformal Killing-Yano tensor.

Introduction.

The Hertz potential scheme has proven useful for solving Maxwell's equations in flat space, and has also been extended to curved spacetimes [6]. In certain algebraically special spacetimes, a further reduction is possible, since the components of a Hertz potential can be expressed in terms of a complex function satisfying a scalar wave equation. Such a function is known as a *Debye potential.* In conformally flat spacetimes, the method of Debye potentials can be applied to massless fields of any spin by means of the *spin raising* operator associated with a twistor. Conversely, a massless spinor field yields a function satisfying the Debye wave equation after sufficient applications of *spin lowering* with a twistor. In general, spin raising and lowering may be used to generate a massless field of any desired spin from another [9].

The purpose of this paper is to show how this technique can be extended to a broader class of spacetimes by raising and lowering with spinors corresponding to null shear-free vector fields. These *shear-free spinors* satisfy an equation which may be considered a generalisation of the twistor equation. In this paper we only consider massless Dirac fields. Applications to fields of higher spin will be given in [1]. The class of spacetimes in which the method has been extended is the generalised Goldberg-Sachs class [10]. This consists of all algebraically special spacetimes in which each repeated principal null direction is aligned with a null shear-free vector field, and vice-versa. In a spacetime of this type, we show that a massless Dirac field may be lowered with a shear-free spinor to give a Debye potential, which may then be raised with another shear-free spinor to produce a new massless Dirac field. Thus we have an operator on spinor fields which generates new solutions of the massless Dirac equation from old ones - a symmetry operator. The

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operator is of first order, and it can be shown to be an R-commuting operator, that is, one whose commutator with the Dirac operator is another operator composed with the Dirac operator. Kamran and McLenaghan have shown that the most general operator of this type is a linear combination of three R-commuting operators which are constructed, respectively, from a conformal Killing vector, the Hodge dual of a conformal Killing vector, and a conformal Killing-Yano tensor of degree 2. The operator we discuss in this paper is essentially the component of the operator presented in [8] depending on a conformal Killing-Yano tensor. The spinorial image of a conformal Killing-Yano tensor is a 2-index Killing spinor. Symmetry operators utilising a Killing spinor have been studied in [7]. A conformal Killing-Yano tensor whose divergence vanishes is called a Killing-Yano tensor. If that is the case then we have an operator which *commutes* with the Dirac operator, and so is a symmetry operator for the *massive* Dirac equation. This has been referred to as a generalised angular momentum operator for the Dirac equation [5].

In Section 1. we briefly review the theory of Hertz potentials in curved spaces using the language of exterior forms, as set out in [6]. In Section 2. we summarise the Petrov classification of the conformal tensor following Thorpe [11]. In particular, we characterise algebraically special spacetimes in a form which will be useful to us. Section 3. introduces spinors as elements of a vector space carrying a representation of a Clifford algebra. The connection between this approach and alternatives such as the Infeld-van der Waerden formalism or the γ-matrix representation may be found in [3]. In Section 4. we discuss the connection between null shear-free vector fields, spinors, and repeated principal directions. Section 5. shows how to construct a solution of the massless Dirac equation out of a function satisfying a modified wave equation, and vice-versa. Section 6. shows how this leads to a symmetry operator for the massless Dirac equation, and gives an expression for the operator in terms of a conformal Killing-Yano tensor constructed from a pair of shear-free spinors.

1. Hertz and Debye potentials for a vacuum Maxwell field.

On a spacetime $(\mathcal{M}, g)$, a *Hertz potential* $\mathcal{H}$ is a 2-form chosen so that

$$\triangle\mathcal{H} \;=\; d\mathcal{P} + d^{*}{*}\mathcal{Q} \tag{1}$$

where d is the exterior derivative on forms, $*$ is the Hodge dual, $d^{*} \equiv {*}d{*}$ is the co-derivative and $\mathcal{P}$ and $\mathcal{Q}$ are arbitrary 1-forms. The Laplace-Beltrami operator $\triangle \equiv -(dd^{*} + d^{*}d)$ is a generalisation of the Laplacian to differential forms. Then for the 2-form F given by

$$F \;=\; dd^{*}\mathcal{H} + d\mathcal{P} \tag{2}$$

we also have

$$F \;=\; -d^{*}d\mathcal{H} - d^{*}{*}\mathcal{Q} \tag{3}$$

by (1), thus the vacuum Maxwell equations $dF = 0$ and $d^{*}F = 0$ are satisfied. Furthermore, if the spacetime is in the generalised Goldberg-Sachs class, then the components of

$\mathcal{H}$ may be expressed entirely in terms of a complex function satisfying a wave equation - a Debye potential. For suitable choices of $\mathcal{P}$ and $\mathcal{Q}$, F can be written using the Debye potential and a conformal Killing-Yano tensor (see [2]). We will show in Section 5. that the massless Dirac equation can also be solved using a Debye potential in spacetimes admitting such a tensor.

2. Petrov classification.

The conformal tensor of a spacetime may be regarded as a self-adjoint operator on the space of complex self-dual 2-forms. The Petrov classification scheme characterises the Jordan canonical form of the conformal tensor C when acting on 2-forms in this way. Given a basis $\{X_a\}$ for vector fields with dual basis $\{e^a\}$, the conformal tensor may be written using a set of *conformal 2-forms* $C^a{}_b$ as

$$C \quad = \quad 2C^a{}_b \otimes e^b \otimes X_a. \tag{4}$$

Putting $C_{ab} = g_{ac}C^c{}_b$, the conformal 2-forms are defined by

$$C_{ab} \quad = \quad R_{ab} - \frac{1}{2}\left(P_a \wedge e_b - P_b \wedge e_a\right) + \frac{1}{6}\mathcal{R} e_a \wedge e_b \tag{5}$$

where R_{ab} are the *curvature 2-forms* derived from the curvature tensor R by

$$R^a{}_b(X,Y) \quad = \quad \frac{1}{2}R(X,Y,X_b,e^a) \quad \forall\, X,Y \in \Gamma T\mathcal{M}. \tag{6}$$

(The symbol $\Gamma T\mathcal{M}$ denotes the space of sections of the tangent bundle of $\mathcal{M}$, that is, vector fields on $\mathcal{M}$). The *Ricci 1-forms* P_a are given by

$$P_a = \mathbf{Ric}(X_b, X_a)e^b \tag{7}$$

and $\mathcal{R} = \mathbf{Ric}(X_a, X^a)$ is the *curvature scalar.*

Using the conformal 2-forms, the action of C on a 2-form ϕ is

$$C \cdot \phi \quad = \quad \frac{1}{2}X_b \rfloor X_a \rfloor \phi C^{ab} \tag{8}$$

where $\rfloor$ denotes the interior derivative on forms. The space of complex self-dual 2-forms has (complex) dimension 3. An algebraically general spacetime has three independent eigenvectors with distinct eigenvalues, all other cases being algebraically special.

A self-dual 2-form ϕ that is also *null*, that is, $\phi \wedge \phi = 0$, is decomposable and so determines a real null vector K such that $K \rfloor \phi = 0$. This K is unique up to scalings. The *principal null directions* of the conformal tensor are the vectors determined by the null self-dual 2-forms satisfying

$$\phi \wedge C \cdot \phi \quad = \quad 0. \tag{9}$$

If an eigenvector of C is null, it determines a *repeated* principal null direction. In fact, algebraically special spacetimes may be characterised as those admitting a null eigenvector. If C has two null eigenvectors, then both of them must have the same eigenvalue and the spacetime must be either type D or conformally flat.

3. Clifford algebras and spinors.

The *spinor space* at a point p of $\mathcal{M}$ is a 4-dimensional complex vector space which carries an irreducible representation of the complex Clifford algebra generated by the cotangent space at p, $T_p^*\mathcal{M}$. Denoting Clifford multiplication by juxtaposition, the Clifford algebra is generated by taking all possible products of vectors in the complexified cotangent space, together with the relations

$$e^a e^b + e^b e^a \;=\; 2g^{ab}. \qquad (10)$$

In four dimensions, the Clifford algebra is isomorphic to the algebra of 4×4 complex matrices. A convenient basis for the Clifford algebra is

$$\{1, e^a, \tfrac{1}{2}[e^a, e^b], e^a z, z\}$$

where $[\ ,\ ]$ is the Clifford commutator and $z = e^0 e^1 e^2 e^3$. In the physical literature, it is usual to use a γ-matrix representation of the Clifford algebra. In that case the covectors e^a are represented by γ^a, the commutator $\frac{1}{2}[e^a, e^b]$ by σ^{ab} and iz by γ^5. The Clifford algebra can also be regarded as the space of complex exterior forms of $T_p^*\mathcal{M}$, with the Clifford product being related to the exterior product and interior derivative by

$$e^a \omega \;=\; e^a \wedge \omega + X^a \rfloor \omega \quad \forall\, \omega \in \Lambda(T_p^*\mathcal{M}). \qquad (11)$$

Thus any form can be considered a Clifford element, and vice-versa. The basis for the exterior algebra corresponding to the Clifford basis given above is

$$\{1, e^a, e^{ab}, *e^a, *1\}$$

where we introduce the abbreviation $e^{ab} = e^a \wedge e^b$.

Since $z^2 = -1$ the spinor space may be split into two 2-dimensional eigenspaces of z with eigenvalues $\pm i$. Eigenspinors of z with eigenvalue $-i$ will be called *even*, while those with eigenvalue $+i$ will be called *odd*. In the following we will primarily be using even spinors, since they may be used to construct self-dual 2-forms (see Section 4.). Spinor space also possesses a $\mathbb{C}$-linear anti-symmetric inner product $(\ ,\ )$. Denoting the action of a form on a spinor by left-multiplication, the inner product has the property that for spinors u and v and a p-form ω,

$$(\omega u, v) \;=\; (-1)^{\lfloor p/2 \rfloor}(u, \omega v). \qquad (12)$$

There is also a representation independent complex conjugate, the *charge conjugate*. The charge conjugate of u is written u^c.

Provided that certain topological conditions are met, we can form a *spinor bundle* over $\mathcal{M}$. The spinor bundle has the property that each fibre over a point $p \in \mathcal{M}$ carries an irreducible representation of the Clifford algebra of $T_p^*\mathcal{M}$. A *spinor field* is then a section of the spinor bundle.

4. Shear-free vector fields and spinors.

A *shear-free vector field* may be characterised as a vector field K which generates a conformal isometry on its conjugate space $K^{\perp} = \{X \in \Gamma T\mathcal{M} : g(X,K) = 0\}$. Note that if K is null then $K \in K^{\perp}$, and in order for the shear of K to be well-defined we require K to be geodesic. If K is null and real then it may be related to an even spinor u by

$$K = (iu^{\mathrm{c}}, e^a u) X_a. \tag{13}$$

Then K is a null shear-free geodesic vector field if and only if

$$(u, \nabla_{X_a} u) e^a u = 0. \tag{14}$$

(See, for example, [4]). A convenient restatement of this equation is that u satisfies

$$\widehat{\nabla}_X u - \frac{1}{4} X^\flat \widehat{\mathrm{D}} u = 0 \quad \forall\, X \in \Gamma T\mathcal{M} \tag{15}$$

where $X^\flat$ is the 1-form such that $X^\flat(Y) = g(Y,X)$ for all $Y \in \Gamma T\mathcal{M}$. We also introduce a $GL(1,\mathbb{C})$-*gauged covariant derivative* $\widehat{\nabla}$ with Dirac operator $\widehat{\mathrm{D}} = e^a \widehat{\nabla}_{X_a}$, where

$$\widehat{\nabla}_X u = \nabla_X u + q\mathcal{A}(X) u \tag{16}$$

$$\widehat{\mathrm{D}} u = \mathrm{D} u + q\mathcal{A} u \tag{17}$$

and $\mathcal{A}$ is a complex 1-form. The action of $\widehat{\nabla}$ on a spinor will depend on the constant q, referred to as the $GL(1,\mathbb{C})$-*charge* of u. In particular, it will reduce to the ordinary covariant derivative when acting on spinors with zero charge. The gauged covariant derivative must also obey a Leibniz rule with respect to Clifford products and the Clifford action on spinors. Thus for a form (or Clifford element) ω with charge q_1 and a spinor u with charge q_2 we have

$$\begin{aligned} \widehat{\nabla}_X(\omega u) &= \widehat{\nabla}_X \omega u + \omega \widehat{\nabla}_X u \\ &= \nabla_X \omega u + q_1 \mathcal{A}(X) \omega u + \omega \nabla_X u + q_2 \mathcal{A}(X) \omega u \\ &= \nabla_X(\omega u) + (q_1 + q_2)\, \mathcal{A}(X) \omega u \end{aligned} \tag{18}$$

so the spinor ωu has charge $q_1 + q_2$. Spinors obeying (15) will be referred to as *shear-free spinors*. The shear-free spinor equation can be considered a $GL(1,\mathbb{C})$-covariant twistor equation. Clearly shear-free spinors with zero charge are twistors.

Differentiating (15) introduces the curvature operator $\hat{R}(X,Y)$ of $\widehat{\nabla}$. This is related to the curvature operator of ∇ by

$$\hat{R}(X,Y) u = R(X,Y) u + q Y \rfloor X \rfloor \mathcal{F} u \tag{19}$$

where $\mathcal{F} = d\mathcal{A}$ is the $GL(1,\mathbb{C})$-curvature. Since the curvature operator on spinors is related to the curvature 2-forms by

$$R(X,Y) u = \frac{1}{2} e^a(X) e^b(Y) R_{ab} u \tag{20}$$

the shear-free spinor equation has the integrability condition

$$R_{ab}u + 2qX_b \lrcorner X_a \lrcorner \mathcal{F}u - \frac{1}{2}(e_b\widehat{\nabla}_{X_a} - e_a\widehat{\nabla}_{X_b})\widehat{\mathrm{D}}u \quad = \quad 0. \tag{21}$$

Clifford multiplication on the left by e^a gives

$$P_b u - 2qX_b \lrcorner \mathcal{F}u + \widehat{\nabla}_{X_b}\widehat{\mathrm{D}}u + \frac{1}{2}\widehat{\mathrm{D}}^2 u \quad = \quad 0. \tag{22}$$

A further multiplication by e^b leads to

$$\mathcal{R}u - 4q\mathcal{F}u + 3\widehat{\mathrm{D}}^2 u \quad = \quad 0. \tag{23}$$

An even spinor u determines a null self-dual 2-form ϕ by

$$\phi \quad = \quad -\frac{1}{8}(u, e_{ab}u)e^{ab}. \tag{24}$$

The real null vector K associated with ϕ is the same as that determined by u in (13). If u is shear-free, then (21)–(23) and (5) show that

$$C_{ab}u \quad = \quad q\left(\frac{1}{6}e_{ab}\mathcal{F} + \frac{1}{2}\mathcal{F}e_{ab}\right)u. \tag{25}$$

Then (24) and (25) show that ϕ satisfies (9), and hence K is a principal null direction. Furthermore, if the spacetime is in the generalised Goldberg-Sachs class, K must be a *repeated* principal null direction and thus ϕ is an eigenvector of the conformal tensor. If $C \cdot \phi = \mu\phi$ for a complex function μ then from (24) and (25) it can be shown that

$$q\mathcal{F}u \quad = \quad -3\mu u. \tag{26}$$

5. Debye potentials for the massless Dirac equation.

Let f be a complex scalar field with $GL(1,\mathbb{C})$-charge -1, that is,

$$\widehat{\nabla}_X f \quad = \quad \nabla_X f - \mathcal{A}(X)f. \tag{27}$$

Given an even shear-free spinor u with charge $+1$, we can construct a $GL(1,\mathbb{C})$-neutral spinor σ by taking

$$\sigma \quad = \quad \hat{d}fu + \frac{1}{2}f\widehat{\mathrm{D}}u \tag{28}$$

where

$$\begin{aligned} \hat{d}f \quad &= \quad e^a \wedge \widehat{\nabla}_{X_a}f \\ &= \quad df - f\mathcal{A} \end{aligned} \tag{29}$$

is the gauged exterior derivative of f. Note that $\hat{d}^2 f = -f\mathcal{F}$. Acting on σ with the Dirac operator produces

$$\begin{aligned} \mathrm{D}\sigma &= e^a\left(\widehat{\nabla}_{X_a}\hat{d}f.u + \hat{d}f\widehat{\nabla}_{X_a}u + \frac{1}{2}\widehat{\nabla}_{X_a}f.\widehat{\mathrm{D}}u + \frac{1}{2}f\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u\right) \\ &= \hat{d}^2 fu - \hat{d}^*\hat{d}fu + e^a\hat{d}f\widehat{\nabla}_{X_a}u + \frac{1}{2}\hat{d}f\widehat{\mathrm{D}}u + \frac{1}{2}f\widehat{\mathrm{D}}^2u \\ &= -f\mathcal{F}u - \hat{d}^*\hat{d}fu + \frac{1}{2}f\widehat{\mathrm{D}}^2u + 2\left(\widehat{\nabla}_{\widehat{\mathrm{grad}}f}u - \frac{1}{4}\hat{d}f\widehat{\mathrm{D}}u\right) \end{aligned} \tag{30}$$

where $\widehat{\mathrm{grad}}f$ is the dual of $\hat{d}f$. Defining the gauged Laplace-Beltrami operator by

$$\hat{\triangle} = -(\hat{d}^*\hat{d} + \hat{d}\hat{d}^*) \tag{31}$$

where $\hat{d}^* = *\hat{d}*$ we have $\hat{\triangle}f = -\hat{d}^*\hat{d}f$. Applying the shear-free condition (15) and the integrability condition (23) to (30) we have

$$\begin{aligned} \mathrm{D}\sigma &= \hat{\triangle}fu - f\mathcal{F}u + \frac{1}{2}f\widehat{\mathrm{D}}^2u \\ &= \hat{\triangle}fu - f\mathcal{F}u + \frac{1}{2}f\left(-\frac{1}{3}\mathcal{R}u + \frac{4}{3}\mathcal{F}u\right) \\ &= \left(\hat{\triangle}f - \frac{1}{6}\mathcal{R}f\right)u - \frac{1}{3}f\mathcal{F}u. \end{aligned} \tag{32}$$

If the spacetime is in the generalised Goldberg-Sachs class then by (26) we have $\mathcal{F}u = -3\mu u$, so

$$\mathrm{D}\sigma = \left(\hat{\triangle}f - \frac{1}{6}\mathcal{R}f + \mu f\right)u. \tag{33}$$

Thus σ is a solution of the massless Dirac equation whenever f satisfies

$$\hat{\triangle}f - \frac{1}{6}\mathcal{R}f = -\mu f. \tag{34}$$

Conversely, given a second shear-free spinor with opposite charge to u and a solution of the massless Dirac equation, their inner product is a function satisfying (34). Let v be an even shear-free spinor with charge -1. Then for an arbitrary spinor ψ the function $f = (v,\psi)$ has charge -1. The action of the gauged Laplace-Beltrami operator on f is

$$\begin{aligned} \hat{\triangle}f &= (\widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a}v,\psi) + (\widehat{\nabla}_{X^a}v,\nabla_{X_a}\psi) + (\widehat{\nabla}_{X_a}v,\nabla_{X^a}\psi) \\ &\quad +(v,\nabla_{X_a}\nabla_{X^a}\psi) - (\widehat{\nabla}_{\nabla_{X_a}X^a}v,\psi) - (v,\nabla_{\nabla_{X_a}X^a}\psi) \\ &= (\widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a}v - \widehat{\nabla}_{\nabla_{X_a}X^a}v,\psi) + 2(\widehat{\nabla}_{X^a}v,\nabla_{X_a}\psi) \\ &\quad +(v,\nabla_{X_a}\nabla_{X^a}\psi - \nabla_{\nabla_{X_a}X^a}\psi). \end{aligned} \tag{35}$$

Now the square of the gauged Dirac operator may be written as

$$\widehat{\mathrm{D}}^2\psi = \widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a}\psi - \widehat{\nabla}_{\nabla_{X_a}X^a}\psi - \frac{1}{4}\mathcal{R}\psi + q\mathcal{F}\psi \tag{36}$$

and so

$$\hat{\triangle} f = (\widehat{\mathrm{D}}^2 v + \frac{1}{4}\mathcal{R}v + \mathcal{F}v, \psi) + 2(\widehat{\nabla}_{X^a} u, \nabla_{X_a}\psi) + (v, \mathrm{D}^2\psi + \frac{1}{4}\mathcal{R}\psi). \quad (37)$$

By (12), (15) and (23) we have

$$\begin{aligned}\hat{\triangle} f &= (-\frac{1}{3}\mathcal{R}v - \frac{4}{3}\mathcal{F}v + \frac{1}{4}\mathcal{R}v + \mathcal{F}v, \psi) \\ &\quad + \frac{1}{2}(e^a \widehat{\mathrm{D}} v, \nabla_{X_a}\psi) + (v, \mathrm{D}^2\psi + \frac{1}{4}\mathcal{R}\psi) \\ &= \frac{1}{6}\mathcal{R}f - \frac{1}{3}(\mathcal{F}v, \psi) + \frac{1}{2}(\widehat{\mathrm{D}}v, \mathrm{D}\psi) + (v, \mathrm{D}^2\psi). \quad (38)\end{aligned}$$

Using the fact that u and v are shear-free spinors with opposite charge, we can deduce from (25) that the two null 2-forms corresponding to u and v are eigenvectors of the conformal tensor with the same eigenvalue. Thus a spacetime admitting a pair of shear-free spinors with opposite charge is necessarily type D and simultaneously in the generalised Goldberg-Sachs class. Equation (26) must also hold for v, so $\mathcal{F}v = 3\mu v$. Then

$$\hat{\triangle} f - \frac{1}{6}\mathcal{R}f = -\mu f + \frac{1}{2}(\widehat{\mathrm{D}}v, \mathrm{D}\psi) + (v, \mathrm{D}^2\psi) \quad (39)$$

thus f is a solution of (34) whenever $\mathrm{D}\psi = 0$.

We define a self-dual (but non-null) 2-form ω by

$$\omega = -\frac{1}{8}(u, e_{ab}v)e^{ab}. \quad (40)$$

It can be shown that u and v are shear-free spinors with opposite charge if and only if ω satisfies the equation

$$3\nabla_X \omega = X \lrcorner d\omega - X^\flat \wedge d^*\omega. \quad (41)$$

Solutions of this equation are known as *conformal Killing-Yano tensors*. The spinorial counterpart of this object is known as a *valence 2 Killing spinor*. The Killing spinor κ associated with ω may be written as

$$\kappa = \frac{1}{2}(u \otimes v + v \otimes u). \quad (42)$$

6. Symmetry operators for the massless Dirac equation.

The operations described in Section 5. provide a scheme for producing new solutions of the massless Dirac equation from old ones. Given an arbitrary spinor ψ and a pair of shear-free spinors u and v with opposite charge, the function $f = (v, \psi)$ satisfies (39). So for σ as defined in (28) we have

$$\begin{aligned} \mathrm{D}\sigma &= \left(\hat{\triangle} f - \frac{1}{6}\mathcal{R} f + \mu\right) u \\ &= \left(\frac{1}{2}(\widehat{\mathrm{D}} v, \mathrm{D}\psi) + (v, \mathrm{D}^2\psi)\right) u \end{aligned} \tag{43}$$

hence $\mathrm{D}\sigma = 0$ whenever $\mathrm{D}\psi = 0$. It is possible to express this as a *symmetry operator* for D, that is, an operator which maps the kernel of D into itself. The gauged covariant derivative is compatible with the spinor inner product, so

$$\begin{aligned} \hat{d} f &= (\widehat{\nabla}_{X_a} v, \psi) e^a + (v, \nabla_{X_a}\psi) e^a \\ &= \frac{1}{4}(e_a \widehat{\mathrm{D}} v, \psi) e^a + (v, \nabla_{X_a}\psi) e^a \end{aligned} \tag{44}$$

thus,

$$\sigma = \frac{1}{4}(e_a \widehat{\mathrm{D}} v, \psi) e^a u + (v, \nabla_{X_a}\psi) e^a u + \frac{1}{2}(v, \psi)\widehat{\mathrm{D}} u. \tag{45}$$

This may be written using the conformal Killing-Yano tensor ω defined in (40) as

$$\sigma = e^a \omega \nabla_{X_a}\psi + \frac{2}{3} d\omega\psi - \frac{2}{3} d^*\omega\psi - \frac{[\phi_1, \phi_2]\cdot\omega}{4\omega\cdot\omega}\mathrm{D}\psi. \tag{46}$$

In this expression ϕ_1 and ϕ_2 are the 2-forms corresponding to u and v respectively via (24), and the symbol $\cdot$ is the inner product on 2-forms defined by

$$\phi\cdot\psi = -*(\phi\wedge *\psi) \quad \forall\ \phi, \psi \in \Gamma\Lambda^2\mathcal{M}. \tag{47}$$

Note that the Clifford commutator of two 2-forms is also a 2-form. Then (43) and (46) show that

$$\mathrm{D}\left(e^a \omega \nabla_{X_a}\psi + \frac{2}{3} d\omega\psi - \frac{2}{3} d^*\omega\psi\right) = 0 \tag{48}$$

whenever $\mathrm{D}\psi = 0$. The operator $\mathcal{K}_\omega$ defined by

$$\mathcal{K}_\omega = e^a \omega \nabla_{X_a} + \frac{2}{3} d\omega - \frac{2}{3} d^*\omega \tag{49}$$

must then be a symmetry operator for the massless Dirac equation.

While in this instance, ω is specifically self-dual and non-null, a straightforward calculation utilising (41) and it's derivatives shows that

$$[\mathrm{D}, \mathcal{K}_\omega] = \left(\omega\mathrm{D} - \frac{1}{3} d\omega + d^*\omega\right)\mathrm{D} \tag{50}$$

from which it is clear that $\mathcal{K}_\omega$ is a symmetry operator, without requiring ω to be self-dual or non-null. Operators such as $\mathcal{K}_\omega$ whose commutator with D is of the form $R\mathrm{D}$ (where R is another operator) are called *R-commuting.*

A slight modification of $\mathcal{K}_\omega$ leads to what has been referred to as a generalised total angular momentum operator for the Dirac equation [5]. Putting

$$\mathcal{L}_\omega = z\left(e^a\omega\nabla_{X_a} + \frac{2}{3}d\omega - \frac{2}{3}d^*\omega - \omega\mathrm{D}\right) \tag{51}$$

it can be verified that

$$[\mathrm{D}, \mathcal{L}_\omega] = -\frac{2}{3}zd^*\omega\mathrm{D} \tag{52}$$

whenever ω is a conformal Killing-Yano tensor. This is essentially the component of the general R-commuting operating found in [8] which is constructed from a conformal Killing-Yano tensor. We observe that $\mathcal{L}_\omega$ commutes with D when $d^*\omega = 0$, in which case ω is called a *Killing-Yano tensor*. When $d^*\omega = 0$ the operator $\mathcal{L}_\omega$ reduces to that found in [5], and $\mathcal{L}_\omega$ is a symmetry operator for the *massive* Dirac equation.

7. Conclusion.

We have shown that in a spacetime which admits a pair of shear-free spinors with opposite charge, solutions of the massless Dirac equation may be found by solving a complex scalar wave equation. Such spacetimes must be Petrov type D and in the generalised Goldberg-Sachs class, or else conformally flat. Furthermore, since solutions of the massless Dirac equation can be used to generate solutions of the required scalar equation, we have a symmetry operator for the massless Dirac equation which may be written in terms of the non-null self-dual conformal Killing-Yano tensor determined by the shear-free spinors. Taking the commutator of this operator with the Dirac operator shows that satisfying the conformal Killing-Yano equation is sufficient for the operator to be a symmetry operator. In [1] we present a generalisation of this operator to all dimensions and signatures, using a conformal Killing-Yano tensor of arbitrary degree. While the existence of a conformal Killing-Yano tensor places severe constraints on a spacetime, Penrose and Walker [12] have demonstrated that every vacuum type D spacetime admits a valence 2 Killing spinor, which is equivalent to the existence of a conformal Killing-Yano tensor.

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References

[1] I M Benn, Philip Charlton, and Jonathan Kress. Debye potentials for Maxwell and Dirac fields from a generalisation of the Killing-Yano equation. University of Newcastle preprint, gr-qc/9610037.

[2] I M Benn and Jonathan Kress. Force-free fields from Hertz potentials. *Journal of Physics A: Mathematical and General*, 29:6295–6304, 1996.

[3] I M Benn and R W Tucker. *An Introduction to Spinors and Geometry with Applications in Physics*. IOP Publishing Ltd, Bristol, 1987.

[4] H Bondi, A Trautman, and F A E Pirani. *Lectures on General Relativity*. Prentice-Hall, Englewood Cliffs, 1965.

[5] B Carter and R G McLenaghan. Generalized total angular momentum operator for the Dirac equation in curved space-time. *Physical Review D*, 19(4):1093–1097, 1979.

[6] J M Cohen and L S Kegeles. Electromagnetic fields in curved spaces: A constructive procedure. *Physical Review D*, 10(4):1070–1084, 1974.

[7] G F Torres del Castillo. Killing spinors and massless spinor fields. *Proceedings of the Royal Society of London A*, 400:119–126, 1985.

[8] N Kamran and R G McLenaghan. Symmetry operators for the neutrino and Dirac fields on curved spacetime. *Physical Review D*, 30(2):357–362, 1984.

[9] R Penrose and W Rindler. *Spinors and Space-time*, volume 2. Cambridge University Press, Cambridge, 1986.

[10] I Robinson and A Schild. Generalization of a theorem by Goldberg and Sachs. *Journal of Mathematical Physics*, 4(4):484–489, 1963.

[11] J A Thorpe. Curvature and Petrov canonical forms. *Journal of Mathematical Physics*, 10(1):1–7, 1969.

[12] M Walker and R Penrose. On quadratic first integrals of the geodesic equations for the type {22} spacetimes. *Communications in Mathematical Physics*, 18:265–274, 1970.