# On the key functions of axisymmetric gravitation

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**Abstract**

Time-dependent solutions of Einstein's vacuum field equations for axially symmetric non-rotating sources can be constructed exactly from key functions, which themselves are solutions of a certain key partial differential equation in two independent variables. Special examples of key functions (along with their consequential solutions of the field equations) are readily obtainable, but key functions describing physical models can only be obtained by solving a general problem.

The most promising approach for solving this general problem is to dispense with the parameter that appears in the key equation, thus enabling this third-order equation to be integrated to a second-order one, and introducing thereby an explicit arbitrary function into the equation. This function will need to be chosen in a model dependent manner.

## 1. The key equation in third-order form

The talk at the Conference began with a review of those procedures for solving Einstein's vacuum field equations, in situations where there is axial symmetry and no rotation, that have been developed in three papers (Waylen 1982, 1987, 1993). It was shown in outline how the general solution of the field equations can be expressed as an axial expansion (valid near the axis of symmetry) and how, for the particular circumstance of a time-dependent similarity solution (*à la* Lie), this expansion may then be summed to give a closed, and therefore exact expression.

All time-dependent similarity solutions of the field equations share the property that they can be constructed from a key function $\chi(t, z)$. This will have a different functional value for each of these solutions, where the values which $\chi$ is allowed are the solutions of the key partial differential equation

$$\begin{aligned} 4\chi_{,tt}\chi_{,tz}\chi_{,zz} \quad &+ \quad (\chi_{,t} - 2Kz)[\chi_{,tt}\chi_{,zzz} - \chi_{,zz}\chi_{,ttz}] \\ &+ \quad (\chi_{,z} + 2Kt)[\chi_{,zz}\chi_{,ttt} - \chi_{,tt}\chi_{,tzz}] = 0. \end{aligned}$$

During the talk, several examples of special solutions of this equation were presented, all of them producing time-dependent, but otherwise quite uninteresting special similarity solutions of the field equations.

## 2. The key equation in second-order form

In order to look for exact similarity solutions capable of describing sensible physical models, one may elect to simplify the above third-order equation by setting its parametric constant $K = 0$. The resulting key equation can then be integrated to give the second-order equation

$$(\sigma + G)\chi_{,t}^2\chi_{,zz} + (\sigma - G)\chi_{,z}^2\chi_{,tt} = 0,$$

where

$$\sigma = (\chi_{,t}^2\chi_{,z}^2 + G^2)^{1/2}.$$

This elliptic equation is seen to involve a function $G(\chi)$. This is an arbitrary function with one argument, being one of the three such functions that must be involved in the *general* solution for the key function $\chi$.

For convenience below, we shall sometimes write

$$G = 2k\chi/H(\chi).$$

All solutions of the preceding second-order equation will, for the special choice $H(\chi) = 1$, lead to $V(t,z) = 0$ (in the notation of Waylen 1993), and therefore a particular solution is given by $\chi = k(z^2 - t^2)$, which is the standard key function of flat space-time.

The form of the equation that is obtained when one makes the (obviously nonphysical) choice $H(\chi) = 3k\chi^{1/2}/t_0^2$ has $\chi = z^{2/3}(t^2 + t_0^2)$ as a particular solution. This $\chi$ produces the following solution of the field equations (*cf.* Waylen 1993a):

$$\begin{aligned} A &= 3z[(t^2 + t_0^2)Q - t^2P]/[t_0^2(t^2 + t_0^2)^{1/2}], \\ C &= \tfrac{1}{3}(t^2 + t_0^2)^{1/2}[(t^2 + t_0^2)P - t^2Q]/(zt_0^2), \\ B &= t(t^2 + t_0^2)^{1/2}(Q - P)/t_0^2, \end{aligned}$$

where

$$P = 2(3Q^2 - 2)^{3/4}/[Q^{1/2}(3Q^2 - 1)],$$

and $Q$ is to be determined from

$$Q(Q^2 - 1) = \frac{1}{6}t_0^2\rho^2/[z(t^2 + t_0^2)^{3/2}],$$

so that we have $Q = 1$ along the axis $\rho = 0$.
(The comparative simplicity of this solution makes it useful for quick checks.)

## 3. Physical models

One possible choice for a physical model – one that is deduced from Schwarzschild's solution – would be $H(\chi) = 1 + 2m|k/\chi|^{1/2}$. This would still allow us the freedom, in the solution for $\chi$, to make appropriate choices for the two remaining arbitrary functions. If one could thus determine a suitable $\chi$, such that its corresponding similarity solution of the field equations possessed acceptable global (and not merely axial) asymptotic properties, one might then be motivated to search also for other interesting exact models, such as a two-body model. One would probably proceed by first choosing $H(\chi)$ in line with the values which are assumed by Newtonian gravitational potentials along the axis of symmetry.

## Appendix

We note here that the above second-order version of the key partial differential equation (with $K = 0$) constitutes the integrability condition for the following pair of equations for a function $X(t, z)$ :

$$\begin{aligned} X_{,t} &= \frac{\alpha}{G\chi_{,z}}(\sigma + G) + \beta\chi_{,t}[ln(\sigma + G) - 2ln\chi_{,z}], \\ X_{,z} &= \frac{\alpha}{G\chi_{,t}}(\sigma - G) + \beta\chi_{,z}[ln(\sigma + G) - 2ln\chi_{,z} + 2], \end{aligned}$$

where $\alpha$ and $\beta$ are parameters.

## References

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